Exceptional Points of Degeneracy in a Transmission Line Periodically Loaded with Gain and Radiation Loss

Ehsan Hafezi, Ahmed F. Abdelshafy, Tarek Mealy, Alireza Nikzamir, Filippo Capolino
Department of Electrical Engineering and Computer Science
University of California Irvine
Irvine, CA, USA
{ehafezia, abdelsha, tmealy, anikzami, f.capolino}@uci.edu

***Abstract*—We demonstrate that a periodic transmission line consisting of uniform lossless segments together with discrete gain and radiation-loss elements supports exceptional points of degeneracy (EPDs). We provide analytical expressions for the conditions that guarantee the coalescence of eigenvalues and eigenvectors. We show the dispersion diagram and discuss the tunability of the EPD frequency. Additionally, a special case is shown where the eigenvectors coalesce for all frequencies when a specific relationship between transmission line characteristic impedance, and gain/loss elements holds; in other words, in this situation, exceptional points merge to a line of frequency. The class of EPDs proposed in this work is very promising in many of applications that incorporate radiation losses.**

I. INTRODUCTION

Exceptional points of degeneracy (EPDs) are points in parameter space that describe a strong degeneracy in an electromagnetic system. At the EPD, two or more eigenstates of the system coalesce into a single degenerate eigenstate. Due to this fact, 'D' is used to stress the importance of the degeneracy of eigenvectors and not only of eigenvalues [1]. The number of degenerated eigenstates is referred to the order of the exceptional point. In proximity of an EPD, eigenvalues associated to the coalescing eigenvectors change with respect to frequency as $(\omega-\omega_e)\propto(\lambda-\lambda_e)^m$, in which λ_e, ω_e and m are the degenerate eigenvalue, EPD angular frequency, and order of EPD respectively.

Exceptional points have been found in systems satisfying parity-time symmetry [2], in lossless waveguides [3], and also in time-varying systems [4, 5]. The EPD phenomenon has been proved to have a wide range of applications, including high quality factor (Q) and low threshold lasers [6], single-mode operating lasers [7], etc. Moreover, the deviation of the perturbed eigenvalues from the degenerate eigenvalue is large when a small perturbation to a system parameter is applied; this level of sensitivity brings another class of applications in sensors [8]. There are a few kinds of structures that exhibit EPD: Periodic lossless waveguides [9], structures with balanced gain and loss [10], and uniform transmission lines (TLs) with proper dispersion [11]. In this work, we will consider a periodic structure with uniform TLs together with elements of gain and loss.

II. TRANSMISSION LINE FORMULATION AND EPD

We consider the simple TL periodically loaded with shunt gain and loss elements shown in Fig. 1. An analogous formulation can be easily obtained for the case when gain and loss are series elements.

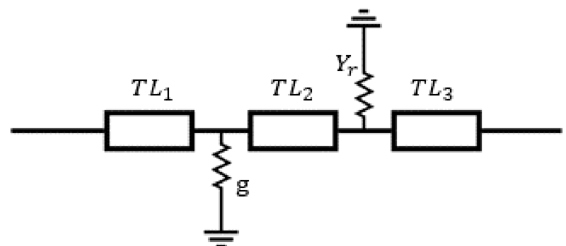

Fig. 1. Unit cell of a periodic transmission line (TL) made of three segments and loaded with shunt lossy (Y_r) and gain (g) elements.

We divide the unit cell into five distinct parts (for simplicity, lines are assumed to have similar characteristic impedance, but with possibly different electrical lengths). Using the transfer matrix of a shunt element and lossless TL, we form a relation between voltage-current between two sides of the unit cell as $\mathbf{\Psi}_{n+1}=\underline{\mathbf{M}}\mathbf{\Psi}_n$, in which the state vector is defined as $\mathbf{\Psi}_n=[V_n \;\; I_n]^t$, with t indicating the transpose action. Furthermore the unit cell transfer matrix $\underline{\mathbf{M}}$ is the result of the multiplication of five transfer matrices:

$$\underline{\mathbf{M}}=\underline{\mathbf{M}}_{TL3}\underline{\mathbf{M}}_{Yr}\underline{\mathbf{M}}_{TL_2}\underline{\mathbf{M}}_{g}\underline{\mathbf{M}}_{TL1}\,. \qquad (1)$$

Note that here we use *forward* transfer matrices, commonly used in various disciplines, which are just the inverse of backward ABCD transfer matrices. We look for solutions satisfying the Floquet's condition $\mathbf{\Psi}_{n+1}=e^{-jkd}\mathbf{\Psi}_n$, where d is the TL period, and we implicitly assume the $e^{j\omega t}$ time convention. This leads to an eigenvalue problem in the form of $[\underline{\mathbf{M}}-e^{-jkd}\underline{\mathbf{I}}]\mathbf{\Psi}_n=\mathbf{0}$, where $\underline{\mathbf{I}}$ is the identity matrix of order two. Eigenvalues $\lambda=e^{-jkd}$ are found by finding the roots of the characteristic polynomial

$$\lambda^2 + [-2\cos(\theta_1+\theta_2+\theta_3) + gY_r Z_0^2 \sin(\theta_2)\sin(\theta_1+\theta_3) - jZ_0 Y_r(1+g/Y_r)\sin(\theta_1+\theta_2+\theta_3)]\lambda + 1 = 0. \quad (2)$$

where θ_i is the electrical length of i^{th} TL segment at the frequency of the interest (i.e., the EPD frequency f_e), and Z_0 is the TL characteristic impedance, assumed to be the same for all TL segments. To have two identical roots in a 2nd order polynomial of the form $\lambda^2 + a\lambda + b = 0$, $a^2 - 4b$ must be zero. Forcing this condition on the coefficients of the characteristic polynomial results in

$$\theta_1+\theta_2+\theta_3 = p\pi \text{, where } p \text{ is an integer number,} \quad (3)$$

$$g = -4/(Z_0^2 Y_r \sin^2(\theta_1+\theta_3)). \quad (4)$$

If these two conditions are met, the two roots will be equal with $e^{-jkd} = (-1)^p$. Fig. 2 depicts an example with $\theta_1 = 45°$, $\theta_2 = 90°$, $\theta_3 = 45°$ at f = 3 GHz and $Y_r = 20$ mS. If a specific EPD frequency is demanded by design, the electrical lengths should be selected at that f_e . Coalescence of the eigenvalues is only a necessary condition for EPDs, therefore we need to show that also the eigenvectors of the system coalesce. The two eigenvectors are found analytically as $\left[jZ_0\left(Y_r - g \pm j\sqrt{Y_r g\left(Y_r g Z_0^2+4\right)}\right)/\left(Y_r+g-jY_r g Z_0\right) \quad 1\right]^t$.

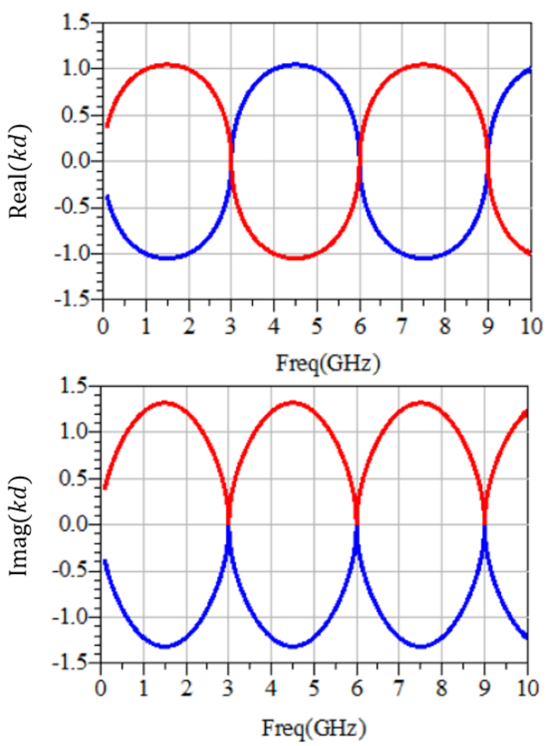

Fig. 2. Dispersion diagram of complex wavenumber versus frequency. Wavenumber degeneracies are observed at 3 GHz, 6 GHz, etc. where both wavenumbers vanish. The two wavenumbers are denoted by different colors.

It is clear that if conditions (3) and (4) are met, the two eigenvectors coalesce to a single eigenvector at the EPD frequency. Upon analyzing the characteristic polynomial, it can be proved that besides the two previously mentioned conditions, when the two extra conditions $Y_r Z_0 = 2$, and $2\theta_1 = 2\theta_3 = \theta_2$ for an arbitrary frequency are met, then the two eigenvalues (and also the eigenvectors) will be identical at every frequency.

III. Conclusion

We have proved theoretically and showed numerically that the periodic TL in Fig. 1 exhibits EPDs. The discrete lossy admittance considered in this paper represents the input admittance of an antenna, which from the TL point of view acts as a loss. We have shown that EPDs occur at frequencies where the two TL wavenumbers vanish, leading to possible applications of broadside radiation in arrays of antennas with elements connected as in the TL in Fig. 1. Such a phenomenon can be used in traveling wave antennas and also in array antennas with all elements oscillating and synchronized.

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