

Two-Dimensional Radiative Transfer Equation Solved Using a Discontinuous Galerkin Method with Mesh Adaptivity

Md Ershadul Haque⁽¹⁾, Hang Wang⁽²⁾, Reza Abedi⁽¹⁾, and Saba Mudaliar⁽³⁾

- ⁽¹⁾ Department of Mechanical Aerospace and Biomedical Engineering, University of Tennessee Space Institute, Manchester, TN 37388-9700, USA
⁽²⁾ Department of Mechanical and Industrial Engineering, New Jersey Institute of Technology, Newark, NJ 07102-1824, USA
⁽³⁾ Sensors Directorate, Air Force Research Laboratory, Wright-Patterson AFB, Dayton, OH 45433-7318, USA, (saba.mudaliar@us.af.mil)

Abstract—The space-angle discontinuous Galerkin (saDG) finite element method is applied to solve the radiative transfer equation (RTE) in two-dimensional enclosures using adaptive mesh refinement (AMR). The discontinuous Galerkin (DG) method is used to discretize both spatial and angular variables, providing a highly flexible and accurate method for modeling radiative heat transfer. The primary focus is on evaluating the impact of mesh adaptivity on the spatial distribution of radiative intensity, ensuring enhanced resolution in regions demanding higher accuracy. Error estimation is guided by the L_∞ -norm, which effectively identifies areas of steep gradients. The saDG method with AMR enhances computational efficiency and accuracy, effectively resolving complex radiative intensity fields in enclosures. This methodology is shown to be a robust and efficient tool for addressing the challenges of radiative heat transfer in two-dimensional geometries.

I. INTRODUCTION

The radiative transfer equation (RTE) is crucial for modeling radiative heat transfer in two-dimensional systems. However, its complexity—stemming from high dimensionality and intricate interactions between radiation and matter—poses significant computational challenges. Traditional numerical methods often struggle to achieve the precision required for complex geometries and heterogeneous materials while maintaining computational efficiency. The discontinuous Galerkin (DG) method provides an effective solution, offering flexibility, high-order accuracy, and robust handling of irregular domains while preserving local conservation properties. These strengths make the DG method particularly well-suited for solving the RTE in complex two-dimensional enclosures [1]. Adaptive mesh refinement (AMR) is a key technique for efficiently solving the RTE, particularly in two-dimensional domains. The RTE frequently involves sharp gradients and localized phenomena that demand high spatial resolution to be captured accurately. A uniformly fine mesh across the entire domain, however, leads to excessive computational costs, making it impractical for complex problems. AMR addresses this by dynamically refining the mesh in regions of high variability while using coarser grids in less critical areas, thus optimizing computational efficiency without compromising accuracy. Jessee *et al.* [2] work on AMR for the radiative transport equation introduces an algorithm that dynamically refines the

mesh based on error estimation, optimizing computational efficiency and accuracy for solving complex radiative transfer problems. Shukai and Samuel [3] introduce a fast, memory-efficient hp - adaptive mesh refinement method that refines both the mesh resolution (h) and polynomial order (p) of the solution basis, balancing accuracy and resource use for solving the radiative transfer problems. When combined with parallel implementation, AMR further accelerates computations by distributing workloads across multiple processors, ensuring scalability, and enabling efficient resolution of large, complex RTE problems.

This study employs a space-angle discontinuous Galerkin method with adaptive mesh refinement to solve the RTE for a two-dimensional enclosure.

II. FORMULATION

The steady-state radiative transfer equation (RTE) for two-dimensional Cartesian coordinates in a gray medium, accounting for absorption, emission, and scattering, is expressed as:

$$\zeta \frac{\partial I(x, y, \mu, \varphi)}{\partial x} + \eta \frac{\partial I(x, y, \mu, \varphi)}{\partial y} + \beta I(x, y, \mu, \varphi) = \kappa I_b(x, y) + \frac{\sigma_s}{4\pi} \int_{-1}^1 \int_{-\pi}^{\pi} I(x, y, \mu', \varphi') \Phi(\mu, \varphi, \mu', \varphi') d\mu' d\varphi' \quad (1)$$

where $I(x, y, \mu, \varphi)$ is the radiative intensity at position (x, y) in the x - y plane, propagating along the direction defined by the unit vector, $\hat{s}(x, y, z) = \{\zeta, \eta, \mu\}$. Assuming that z axis is infinity, where $\zeta = \sin \theta \cos \varphi = \sqrt{1 - \mu^2} \cos \varphi$, $\eta = \sin \theta \sin \varphi = \sqrt{1 - \mu^2} \sin \varphi$, $\mu = \cos \theta$, where $\theta \in [0, \pi]$ is the polar angle and $\varphi \in [-\pi, \pi]$ is the azimuthal angle, measured relative to a fixed spatial axis. The coefficients κ , σ_s , and β represent the spatially dependent absorption, scattering, and extinction coefficients, respectively. The term $I_b(x, y)$ is the spatially dependent total black-body radiative intensity. The scattering phase function $\Phi(\mu, \varphi, \mu', \varphi')$ describes the angular redistribution of radiation during scattering.

The space element Q_x is extruded into m slabs at an angle, resulting in $Q_1, Q_2, \dots, Q_m$ space-angle elements. The sample element Q has the boundaries in x direction, ∂Q_x , and boundaries in $\hat{s}$ direction $\partial Q_{\hat{s}}$. In the 2D domain, the space

mesh consists of unstructured triangles, whereas the extrusion process results in a structured rectangular mesh in angle. For an arbitrary element $\mathcal{Q}$, the residuals are

$$\mathcal{R}_{\mathcal{Q}} = \zeta \frac{\partial I}{\partial x} + \eta \frac{\partial I}{\partial y} + \beta I - \kappa I_b - \frac{\sigma_s}{4\pi} \int_{-\pi}^{\pi} \int_{-1}^1 I(x, y, \mu', \varphi') \Phi(\mu, \varphi, \mu', \varphi') d\mu' d\varphi', \quad (2a)$$

$$\mathcal{R}_{\partial\mathcal{Q}} = \hat{\mathbf{s}} \cdot \mathbf{n} (I^* - I), \quad (2b)$$

The target value I^* corresponds to the upstream value along the direction of wave propagation, which is given.

$$I^* = \begin{cases} I, & \hat{\mathbf{s}} \cdot \mathbf{n} > 0 \\ I^{\text{out}}, & \hat{\mathbf{s}} \cdot \mathbf{n} \leq 0 \end{cases}, \quad (3)$$

$$\hat{\mathbf{s}} \cdot \mathbf{n} = \zeta n_x + \eta n_y \quad (4)$$

It is necessary to specify residuals (errors) on the interior and boundary of elements in a DG formulation. By multiplying the RTE by the weight function $\hat{H}$, the weighted residual (WR) of the finite element equations is created. There are other explanations of the DG formulation in [1].

III. NUMERICAL EXAMPLES

This study focuses on the solution of the steady-state radiative transfer equation within a computational domain for a two-dimensional enclosure using the space-angle discontinuous Galerkin method combined with adaptive mesh refinement. The grid structure is dynamically designed based on an estimation of solution error, with highly refined grids concentrated in regions of the domain exhibiting relatively high spatial discretization errors. The algorithm follows an iterative process: first, the discrete RTE is solved to a specified level of convergence on the current grid structure. Next, the solution error is estimated, and a new refined grid structure is generated. The RTE is then solved again on the updated grid. This process is repeated until the error is reduced to an acceptable level, ensuring both accuracy and efficiency in the solution.

The combination of AMR and the DG method enhances computational efficiency while delivering highly detailed and accurate solutions, making it particularly effective for modeling radiative heat transfer in enclosures with complex spatial intensity distributions. Figure 1 illustrates the AMR strategy applied to solve the RTE in a $(1, 1)$ square enclosure, employing a nested hierarchical mesh structure designed to achieve high accuracy in Level 3 refinement zones. The mesh is progressively refined across three levels based on the spatial variability of the radiative intensity $I(x, y)$. Regions with sharp gradients or discontinuities—primarily along the diagonal from the bottom-left ($x = 0, y = 0$) to the top-right ($x = 1, y = 1$)—are designated as Level 3 refinement zones, where the smallest cells are employed to resolve steep intensity variations with high precision. Conversely, regions

with smooth intensity distributions utilize coarser grids at Level 1 or Level 2 refinement, conserving computational resources while maintaining sufficient accuracy.

The Level 3 refinement zones are critical for capturing fine details in the solution, particularly in areas of strong radiation-matter interactions. This nested hierarchical refinement ensures computational efficiency by concentrating resources where they are most needed. When combined with the DG method, which supports non-uniform grids and provides robust local approximations, this approach achieves a highly accurate and efficient solution, making it ideal for solving complex radiative heat transfer problems.

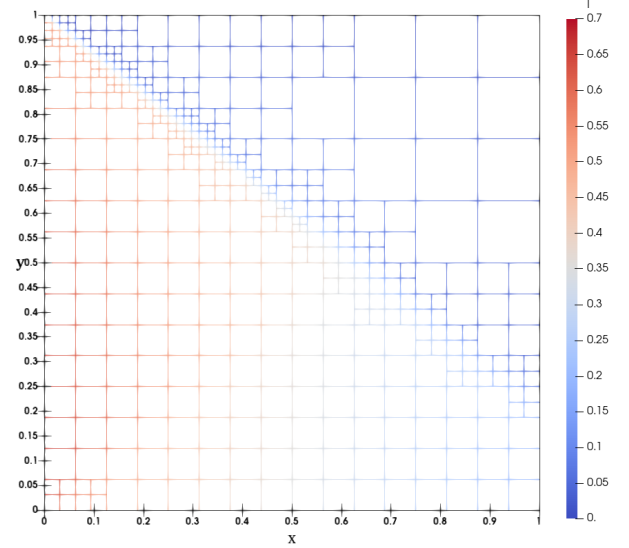


Fig. 1: Adaptive mesh refinement for a square enclosure (three refinement levels).

IV. CONCLUSIONS

We introduced a space-angle DG method for numerically solving the two-dimensional radiative transfer equation by integrating adaptive mesh refinement. The combination of the space-angle discontinuous Galerkin method and adaptive mesh refinement proves highly effective for solving the radiative transfer equation in a square enclosure. By dynamically refining the mesh in regions with sharp intensity gradients, this approach achieves accurate resolution of radiative intensity discontinuities while maintaining computational efficiency.

REFERENCES

- [1] Hang Wang, Md Ershadul Haque, Reza Abedi, and Saba Mudaliar. Integrating angular and domain decomposition with space-angle discontinuous galerkin methods in 2d radiative transfer. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 329:109208, 2024.
- [2] J Patrick Jessee, Woodrow A Fiveland, Louis H Howell, Phillip Colella, and Richard B Pember. An adaptive mesh refinement algorithm for the radiative transport equation. *Journal of computational Physics*, 139(2):380–398, 1998.
- [3] Shukai Du and Samuel N Stechmann. Fast, low-memory numerical methods for radiative transfer via hp-adaptive mesh refinement. *Journal of Computational Physics*, 480:112021, 2023.