

Robust Off-Grid DOA Estimation in Low SNR Using Noise Integral-Based Bayesian Learning

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Abstract—Bayesian learning methods have been applied to the direction of arrival (DOA) estimation. However, algorithm performance can be affected by inappropriate initial values of noise. To address this issue, we propose a noise integral-based inference framework to improve the robustness of the Bayesian method. Additionally, considering the direction mismatch, an off-grid Bayesian method with modified covariance matrix is developed to further improve the performance by iteratively refining the coarse grid. The simulation results show that the proposed scheme outperforms other sparse Bayesian learning (SBL) methods at low signal-to-noise ratio (SNR) and approaches the Cramer-Rao Bound (CRB) at high SNR.

Index Terms—Bayesian learning, DOA estimation, noise integral, off-grid

I. INTRODUCTION

Direction of arrival (DOA) estimation has received increasing attention owing to successful applications in array signal processing. Traditional estimation schemes include multiple signal classification (MUSIC) [1] and estimation of signal parameter via rotational invariance techniques (ESPRIT) [2], whose performance is limited to the number of snapshots. Recently, sparse Bayesian learning (SBL) has been widely used in DOA estimation due to their superior performance.

SBL is based on assuming a sparse prior on the signal of interest and then reconstructing the original signal through expectation-maximization (EM). The true DOA, actually, may not fall on the predefined grid. To solve this mismatch problem, the authors in [3] propose the off-grid sparse Bayesian inference (OGSBI), which introduces the grid mismatch error by the first-order Taylor approximation. However, it is difficult to balance computational complexity and accuracy for OGSBI. Furthermore, the authors in [4] develop the root SBL by treating the sampled locations as adjustable parameters. The methods mentioned above require estimating the noise at each iteration. However, the hand-crafted initial value of noise may degrade the performance of algorithm [5]. To tackle this issue, we propose a noise integral-based SBL framework. Considering the direction mismatch, a off-grid noise integral-based SBL algorithm is further developed.

II. SYSTEM MODEL

Consider K far-field sources impinging on a uniform linear array (ULA) with M sensors, the array output at time t can be written as

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{x}(t), \quad t \in \{1, \dots, T\}, \quad (1)$$

where $\mathbf{y}(t) = [y_1(t), \dots, y_M(t)]^T$, $\mathbf{x}(t) = [x_1(t), \dots, x_K(t)]^T$ and $\mathbf{v}(t) = [v_1(t), \dots, v_M(t)]^T$ are the output data, source signal and white Gaussian noise,

respectively. $\mathbf{A} = [\boldsymbol{\alpha}(\theta_1), \dots, \boldsymbol{\alpha}(\theta_K)]$ consists of K steering vectors, each of which can be expressed as

$$\boldsymbol{\alpha}(\theta_k) = [e^{-j2\pi d_1 \sin(\theta_k)/\lambda}, \dots, e^{-j2\pi d_M \sin(\theta_k)/\lambda}], \quad (2)$$

where $\theta_k, k \in \{1, \dots, K\}$, is DOA of k -th source, and λ is the wave-length. After sampling at the DOA range, the above (1) can be further expressed as

$$\mathbf{y}(t) = \bar{\mathbf{A}}\mathbf{s}(t) + \mathbf{v}(t), \quad t \in \{1, \dots, T\}, \quad (3)$$

where $\bar{\mathbf{A}} = [\boldsymbol{\alpha}(\bar{\theta}_1), \dots, \boldsymbol{\alpha}(\bar{\theta}_N)]$ with $\bar{\theta}_n, n \in \{1, \dots, N\}$, is n -th sampling grid. Let $\mathbf{Y} = [\mathbf{y}(1), \dots, \mathbf{y}(T)]$, $\mathbf{X} = [\mathbf{x}(1), \dots, \mathbf{x}(T)]$ and $\mathbf{V} = [\mathbf{v}(1), \dots, \mathbf{v}(T)]$, the above (3) can be further denoted as

$$\mathbf{Y} = \bar{\mathbf{A}}\mathbf{X} + \mathbf{V}. \quad (4)$$

III. PROPOSED ALGORITHM

In this section, an off-grid noise integral-based sparse Bayesian learning (OGNISBL) is developed for DOA estimation.

A. Noise Integral Based Sparse Bayesian Learning

Under the assumption of white Gaussian noise, the probability density function (PDF) of $\mathbf{Y}$ in (4) can be expressed as

$$p(\mathbf{Y} | \mathbf{X}; \beta_0) = \mathcal{CN}(\bar{\mathbf{A}}\mathbf{X}, \beta_0^{-1}\mathbf{I}_M), \quad (5)$$

where β_0 indicates the noise precision following Gamma distribution

$$p(\beta_0) = \Gamma(\beta_0; \tilde{m}, \tilde{n}), \quad (6)$$

where $\Gamma(\cdot)$ is the Gamma function with $\tilde{m}$ and $\tilde{n}$ denoting small positive constants. We further assume that each row of $\mathbf{X}$ follows complex Gaussian distribution

$$p(\mathbf{X} | \boldsymbol{\gamma}, \beta_0) = \prod_{n=1}^N \mathcal{CN}(\mathbf{X}_n | 0, \beta_0^{-1}\gamma_n \mathbf{I}_T), \quad (7)$$

where $\gamma_n, n \in \{1, \dots, N\}$, denotes the variance of the n -th row of $\mathbf{X}$. Then, the second layer is modeled as Gamma distribution

$$p(\boldsymbol{\gamma}) = \prod_{n=1}^N \Gamma(\gamma_n; 1, \tilde{c}), \quad (8)$$

Based on (5), (6) and (7), we can obtain the posterior probability of $\mathbf{X}$ by integrating β_0 out

$$p(\mathbf{X}|\mathbf{Y};\gamma) = \int p(\mathbf{X}|\mathbf{Y};\gamma, \beta_0) p(\beta_0; \tilde{m}, \tilde{n}) d\beta_0$$

$$= \frac{\Gamma(T+\tilde{m}) \left[\frac{1}{\tilde{n}} (\mathbf{X}-\mu)^H \Sigma^{-1} (\mathbf{X}-\mu) + 1 \right]^{-(T+\tilde{m})}}{\Gamma(\tilde{m}) |\Sigma| (\pi \tilde{n})^T}, \quad (9)$$

where $\mu = \Gamma \bar{\mathbf{A}}^H \Sigma_Y^{-1} \mathbf{Y}$, $\Sigma = \Gamma - \Gamma \bar{\mathbf{A}}^H \Sigma_Y^{-1} \bar{\mathbf{A}} \Gamma$, and $\Sigma_Y = \bar{\mathbf{A}} \Gamma \bar{\mathbf{A}}^H + \mathbf{I}_M$. $\Gamma = \text{diag}(\gamma)$, and Σ_Y is the covariance matrix of array output. Through maximizing $E\{\ln p(\mathbf{Y}, \gamma)\}$, the update for γ_n can be expressed as

$$\gamma_n^{new} = \frac{-T}{2\tilde{c}} + \frac{1}{2\tilde{c}} \sqrt{T^2 + \frac{4\tilde{c} \left[T \Sigma_{nn} + (MT + \tilde{m}) \|\mu_n\|_2^2 \right]}{\sum_{t=1}^T \mathbf{Y}_t^H \Sigma_Y^{-1} \mathbf{Y}_t + \tilde{n}}} \quad (10)$$

B. OGNISBL

Let $[\gamma_{m1}, \dots, \gamma_{mK}]$ are the K largest peak in γ , $[\theta_{m1}, \dots, \theta_{mK}]$ are the corresponding grid points, and θ_{mk}^l and θ_{mk}^r are left and right grid points of θ_{mk} . Let

$$\Sigma_{-k} = \Sigma_Y - \tilde{\mathbf{A}}_k \tilde{\Gamma}_k \tilde{\mathbf{A}}_k^H, \quad (11)$$

where

$$\tilde{\mathbf{A}}_k = \begin{cases} [\alpha(\theta_{mk}), \alpha(\theta_{mk}^r)], & \gamma_{mk}^l \leq \gamma_{mk}^r \\ [\alpha(\theta_{mk}^l), \alpha(\theta_{mk})], & \gamma_{mk}^l > \gamma_{mk}^r \end{cases} \quad (12)$$

$$\tilde{\Gamma}_k = \begin{cases} \text{diag}[\gamma_{mk}, \gamma_{mk}^r], & \gamma_{mk}^l \leq \gamma_{mk}^r \\ \text{diag}[\gamma_{mk}^l, \gamma_{mk}], & \gamma_{mk}^l > \gamma_{mk}^r \end{cases} \quad (13)$$

According to [6], we employ a modified covariance matrix for array output

$$\tilde{\Sigma}_Y = \Sigma_{-k} + \eta_k \alpha(\theta_k) \alpha(\theta_k)^H \quad (14)$$

where η_k is the power of θ_k . Then, we can obtain more accurate estimates

$$\hat{\theta}_k = \arg \max \ln [p(\mathbf{Y}, \gamma)] \Big|_{\Sigma_Y = \tilde{\Sigma}_Y}, \quad \theta_k \in [\theta_{mk}^l, \theta_{mk}^r] \quad (15)$$

IV. SIMULATION RESULTS

In this section, we compare the proposed OGNISBL with OGSBL [3], root-SBL [4], RVM-DOA [6] and the Cramer-Rao Bound (CRB). Considering ULA with $M = 10$ sensors and the root-mean-square-error (RMSE) is used to validate the estimate performance.

Fig. 1 shows the RMSE performance comparison of the different methods versus the SNR. It can be noted that the proposed OGNISBL outperforms other schemes at low SNR (-20 to -5dB). This is attributed to the integral processing of noise, which effectively overcomes the effect of noise. Particularly, the performance of OGNISBL approaches the CRB at high SNR.

Fig. 2 provides the RMSE performance achieved by different methods versus the snapshot number, where the SNR

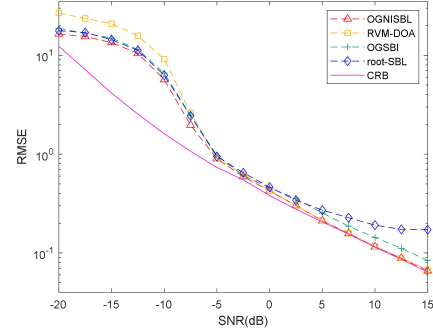


Fig. 1. RMSE performance comparison of different methods versus the SNR.

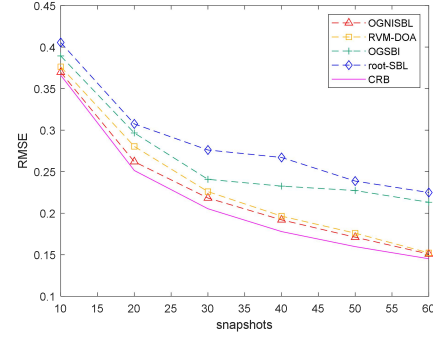


Fig. 2. RMSE performance comparison of different methods versus the snapshot number.

is set to 5dB. It can be seen that the RMSE performance of different methods improves as the number of snapshots increases. Compared with other SBL schemes, the proposed OGNISBL is significantly closer to CRB.

CONCLUSION

In this paper, a off-grid noise integral-based sparse Bayesian learning (OGNISBL) method is proposed for DOA estimation. The simulation results validate that the proposed OGNISBL is superior to other schemes with different SNR and snapshot number.

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