

Practical Aspects of Physical Portrayal of a Green's Function for Poisson's Equation

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Abstract—An apparatus for laboratory visualization of an elementary Green's function of the Poisson's equation with homogeneous boundary condition is extended to accommodate nonhomogeneous boundary conditions. The practical issues encountered and the provisions to meet the requirement of field plotting by current analogy are report in this paper. A variable voltage source with a current sensing resistor is used to emulate a current source and variable resistors terminating a resistive wire are used to generate the voltage references that represent the boundary conditions for the electrostatic problem that the apparatus is designed to characterize. Alignment of the apparatus to reflect the conditions of the electrostatic potential satisfying a one-dimensional Poisson's equation is carried out in an iterative manner. The components for construction of the apparatus can be found in an electronics laboratory and from general hardware suppliers.

I. INTRODUCTION

Green's functions have wide applications in the study of electromagnetic wave propagation, and in the design of antennas and radio frequency circuits and components. Their discussions are found mainly in advanced textbooks and scientific articles whose audience has prior exposure to higher analysis. To facilitate an early introduction of the concept of the Green's function in the curriculum, an elementary problem in electrostatics was invoked to enable a pedagogic construction of a Green's function for the Poisson's equation in one dimensional space and an apparatus was designed to enable a laboratory visualization of the Green's function using field plotting by current analogy. A version of the apparatus to portray a Green's function in one dimension with homogeneous boundary condition was implemented and tested to verify that the voltage along a resistive wire represented a scaled replica of the elementary Green's function. The practical issues in the extension of the apparatus for the more general case with nonhomogeneous boundary condition is the subject of discussion in this paper.

II. PROVISIONS TO ACCOMMODATE NONHOMOGENEOUS BOUNDARY CONDITIONS

The physical setting leading to a solution of Poisson's equation in one spatial dimension is depicted in Fig. 1. It consists of a sheet of charge with uniform density placed between two metal plates held at fixed potentials. For ϕ_0 and ϕ_L are both zero, the Green's function with homogeneous boundary condition was obtained by physical consideration [1]. The more general case of nonhomogeneous boundary condition was treated by

superposition consideration or from first principle based on Green's Theorem [2].

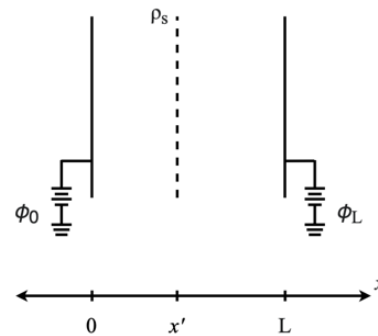


Fig. 1. Setting for an electrostatic problem with a sheet of charge with uniform density placed between two metal plates held at different potentials.

The potential in the space between the metal plates can be written as

$$\phi(x) = \frac{1}{\epsilon_0} \int_0^L G(x, x') \rho(x') dx' + \phi_0 + \frac{\phi_L - \phi_0}{L} x \quad (1)$$

where the Green's function is given by

$$G(x, x') = \frac{(L - x_>)x_<}{L} \quad (2)$$

To enable laboratory visualization of the nature of the Green's function with homogeneous boundary condition, an apparatus based on field plotting with current analogy [3] was designed and tested [4]. The voltage along a resistive wire in the apparatus represents a scaled replica of the potential associated with the electric field in the space between the metal plates held at ground potential.

To incorporate nonhomogeneous boundary conditions, additional voltage sources need to be introduced, leading to the schematic shown in Fig. 2. The potential between the metal plates for the present setting is now a function of three excitations: the sheet of charge, and the potentials the metals plates are held at. Accordingly, the corresponding voltages along the resistive wire in the apparatus are functions of the injected current and the two voltages imposed on the two ends of the resistive wire. The practical considerations for the

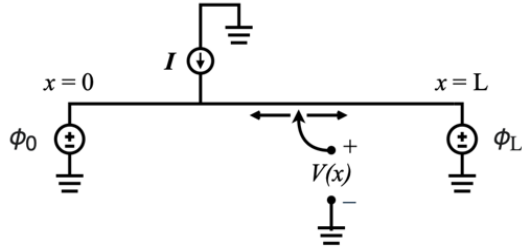


Fig. 2. Schematic of apparatus for visualization of the Green's function for the Poisson's equation with nonhomogeneous boundary conditions depicted by the arrangement in Fig. 1.

arrangement of the current and voltage sources in the apparatus are the subject of discussion in this paper.

While voltage sources or power supplies are readily available as bench equipment, current sources are less common. However, in this application the current source can be emulated by a variable voltage source and a current sensing resistor. For the voltages connected to the two ends of the resistive wire, it may be tempting to employ variable voltage regulators to set the proper boundary conditions. However, most voltage sources operate with current flowing out of its positive terminal, but in the apparatus for this study, the current is actually flowing into the positive terminal of the voltage reference. Most voltage regulators will not function properly under this reversed current condition or may be damaged. Fortunately a simple solution exists if we place a resistor of proper value at each end of the wire so that the voltage drop across the resistor matches with the required voltage reference. Based on these considerations, an implementation of the apparatus can be arrived at as shown in Fig. 3.

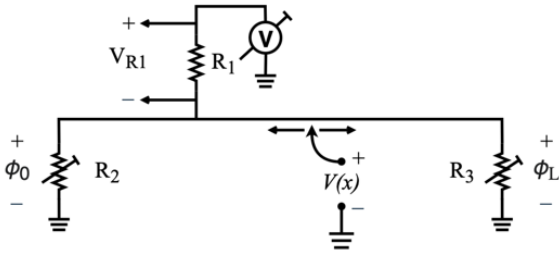


Fig. 3. Practical realization of the schematic shown in Fig. 2 employing variable voltage source and resistors.

The current (chosen to be 0.5A) injected into the resistive wire (36 gauge Nichrome wire with 26 Ω /ft nominal resistivity) is set by monitoring the voltage drop V_{R1} across the current sensing resistor R_1 (5 Ω) in series with the variable voltage source. As the location of current injection is changed along the resistive wire, the voltage source needs to be adjusted to result in the same V_{R1} to ensure the same current is being injected at each location. The two variable resistors (realized by 50 Ω potentiometers) connected to the ends of the resistive wire are adjusted to show the required voltage drop to set the voltage references corresponding to the potentials imposed on the metal plates in the electrostatic problem that the apparatus represents. For each

selected current injection location, all three variables have to be adjusted in an iterative fashion to reach proper values.

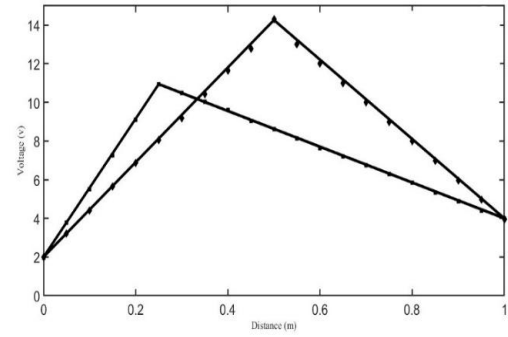


Fig. 4. Measured voltage variations along the resistive wire for current injection at 0.25 m (square datum points) and 0.5 m (rhombic datum points). Theoretical values are shown by the continuous line segments.

Our experience showed that it typically took about 5 to 10 iterations to arrive at the desired setting with less than 3% error. Voltage distributions along the resistive wire for two different current injection locations are shown in Fig. 4. Very good agreement between theory and measurement results is observed.

III. SUMMARY

An apparatus designed to provide a laboratory visualization of an elementary Green's function to provide insight to the characteristics and nature as the response of a physical system to an impulsive excitation was previously implemented for the Poisson's equation in one dimension with homogeneous boundary condition. In the present study, the apparatus is extended to accommodate nonhomogeneous boundary conditions, providing a display of the Green's function for the encompassing integral of the Poisson's equation in one dimension. Practical aspects encountered in the development of the apparatus are described and procedure to meet the physical requirements for field plotting using current analogy realized by commonly available equipment in an electronics laboratory are presented. For each location of current injection on the resistive wire, a specific combination of the voltage of the source and the values of the two variable resistors can be found by iterative method to give the proper current injection and boundary voltages. Typically 5 to 10 iterations can lead to the correct setting with less than 3% error. The apparatus can be constructed with components found in an electronics laboratory and general hardware suppliers.

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