

Quasi-Helmholtz Calderón Multiplicative Preconditioning for Higher-Order Multi-Trace

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Multi-trace integral equations are a powerful tool for modeling scattering of time-harmonic electromagnetic waves on composite objects, as the multi-trace technique readily allows the application of Calderón multiplicative preconditioning to composite structures with junctions. However, if higher-order basis functions, such as the Glisson-Wilton-Peterson (GWP) basis functions (Graglia et al., IEEE Trans. Antennas Propagat., 45, 1997, pp. 329-342), are used in the discretization, traditional Calderón preconditioning is not possible due to the lack of higher-order dual functions, akin to the Buffa Christiansen (BC) functions that are used in conjunction with Rao-Wilton-Glisson RWG in the zeroth-order case.

An alternative Calderón multiplicative preconditioning strategy exists based on the Helmholtz decomposition (Christiansen and Nédélec, SIAM J. Numer. Anal., vol 40, no. 3, pp. 1100-1135, 2002). The discrete Helmholtz decomposition facilitates linking the discretized system to a discretized preconditioner without requiring dual basis functions to have a well-conditioned mixed Gram matrix.

Higher-order quasi-Helmholtz projectors have recently been developed (Bourhis et al., IEEE Trans. Antennas Propagat., vol. 72, no. 4, pp. 3572-3579, 2024). This allows using the alternative Calderón preconditioning strategy with basis functions of arbitrary order. Higher-order quasi-Helmholtz projectors separate solenoidal from non-solenoidal functions. This enables the separate rotation and mapping of solenoidal or nonsolenoidal functions of order p to non-solenoidal or solenoidal functions of order higher than p (Bourhis et al., TechRxiv, 2024). Applying the Calderón multiplicative preconditioner annihilates the square of the discretized hypersingular component of the single-layer operator, guaranteeing the system's well-conditioning.

The computation of the higher-order projectors requires the numerical (pseudo-)inversion of the product of the generalized star matrix. This matrix is similar to the discretization of the Laplacian equation with finite elements of order p . In the case of zeroth-order case, dedicated algorithms like algebraic multigrid can efficiently iteratively solve such systems (Bourhis et al., IEEE Trans. Antennas Propagat., vol. 72, no. 4, pp. 3572-3579, 2024). However, going to higher order makes using standard algebraic multigrid in a black-box fashion less efficient. The quasi-Helmholtz projectors can be recast into an equivalent saddle point problem that can be solved efficiently for higher order.

We use higher-order quasi-Helmholtz projectors to build block diagonal preconditioners for higher-order discretizations of multi-trace formulations. Retaining only the preconditioner's diagonal contributions minimizes the construction cost while still obtaining the desired preconditioning. Numerical experiments corroborate the well-conditioning of the higher-order multi-trace formulation with the quasi-Helmholtz Calderón multiplicative preconditioning (QH-CMP). Fast convergence rates for the QH-CMP are observed regardless of the mesh density and the order of the basis functions when the preconditioning space is two orders higher than the space of the problem.

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