

A Preconditioned Application for the Block Iterative IB-BGMRES-DR Method

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Abstract—Monostatic radar cross section (RCS) calculations often involve solving matrix equations with multiple right-hand sides. Block Krylov subspace methods, particularly Block GMRES, are effective for such problems due to their ability to handle multiple right-hand sides. However, challenges such as algorithmic instability from linearly dependent vectors persist. This paper adopts a block iterative IB-GMRES-DR algorithm, which has the advantages of recycling of spectral information at restart. Based on this IB-GMRES-DR way, we further propose a multi-frontal block incomplete Cholesky (MFBIC) preconditioning technique to enhance its convergence. By using this preconditioning technique, the efficiency of IB-GMRES-DR will be improved a lot. Finally, numerical experiments validate its accuracy and computing performance, which indicates that it has certain potential in some engineering computations.

I. INTRODUCTION

Now, the calculation of monostatic Radar Cross Section (RCS) plays a crucial role in stealth design, radar detection, target recognition, electronic warfare, and theoretical research. Monostatic RCS often involves the calculation of electromagnetic wave scattering in multiple directions. For each incident wave direction or polarization condition, there corresponds a different right-hand-side vector. Therefore, the problem extends from a single right-hand-side vector to a matrix equation with multiple right-hand sides.

For such problems, commonly used methods include direct methods such as LU and QR decomposition, as well as numerical iterative algorithms like GMRES and CG [1]. Although these methods are relatively simple and easy to implement, they can be time-consuming if there are many right-hand sides. Considering that block iterative algorithm can simultaneously solve linear systems with multiple right-hand sides, this approach has seen rapid development. Block Krylov subspace methods are an extended form of Krylov subspace methods, which are designed to solve systems with multiple right-hand sides. The basic iterative steps are modified to handle block matrices and vectors. Besides, block GMRES can be used to solve linear systems regardless of whether they are symmetric or asymmetric, positive-definite or indefinite. Through the orthogonalization process, the block GMRES method can effectively control numerical errors. Thus, the block GMRES method is widely used at present.

During the block iteration process, linearly dependent column vectors may appear in the block residual, which

potentially causes algorithmic instability. To address this issue, R criterion [2] is usually used as the stopping criterion for iterations to ensure that the algorithm terminates when the residual is sufficiently small, thereby guaranteeing the accuracy of the solution. In this paper, we introduce a Block IB-GMRES-DR [2] iterative algorithm to solve matrices with multiple right-hand sides, which has the advantages of recycling of spectral information at restart. Based on IB-BGMRES-DR [2] algorithm, we further propose a preconditioning technique to improve its convergence. In fact, preconditioning techniques often involve matrix inversion operations. Although incomplete Cholesky (IC) decomposition, as a preprocessing technique, can effectively improve the convergence rate of iterative solvers in many scenarios, it may fail to provide sufficiently strong preconditioning for ill-conditioned matrices with high condition numbers, even it could lead to numerical instability. Therefore, in this paper, a multi-frontal block incomplete Cholesky preconditioning (MFBIC) [3] is introduced to improve the IB-BGMRES-DR iterative algorithm. This method utilizes multi-front techniques, and the matrix is broken down into multiple smaller parts. By calculating only partial non-zero matrix elements instead of the entire matrix, this memory usage is further reduced. The rest of the paper is organized as follow. In Section II, a main method on this preconditioner is presented. Section III is numerical example, which the accuracy and the remarkable computational performance are validated.

II. METHOD

In electromagnetic monostatic RCS simulations, the adoption of the IB-BGMRES-DR algorithm often leads to solving a sparse linear matrix equation system, as follow

$$\mathbb{A} \mathbb{X} = \mathbb{B} \quad (1)$$

The dimension of matrix $\mathbb{A}$ is $N \times N$. The matrix $\mathbb{B}$ is typically data generated by external incident electromagnetic waves (such as monostatic incident plane waves or point source excitations), with dimensions of $N \times p$.

Generally speaking, the purpose of preconditioning is to construct a matrix $\mathbb{M}$ such that $\mathbb{M}^{-1}\mathbb{A}$ is close to the identity matrix. To achieve a good preconditioning effect, the preconditioning matrix $\mathbb{M}^{-1}$ needs to be very close to the inverse matrix $\mathbb{A}^{-1}$. However, this construction process often comes at a high cost. Thus, we need dual preconditioners. The

first one is applying hierarchical basis function [4], then combining with the incomplete Choleski factorization, we can get the p-type multigrid preconditioner as follows.

$$\mathcal{M}^{-1} = \begin{bmatrix} I & -(C_{11}^H C_{11})^{-1} A_{12} \\ O & I \end{bmatrix} \begin{bmatrix} (C_{11}^H C_{11})^{-1} & O \\ -(C_{22}^H C_{22})^{-1} A_{21} (C_{11}^H C_{11})^{-1} & (C_{22}^H C_{22})^{-1} \end{bmatrix} \quad (2)$$

Note that there are two matrix inversions about $C_{11}^H C_{11}$ and $C_{22}^H C_{22}$. To further enhance efficiency, we will use MFBIC preconditioning to handle their inversion operations in Eq. (2). Here we take $C_{11}^H C_{11}$ as an example to illustrate.

MFBIC preconditioning technology is similar to the traditional multifrontal way, and it is mainly divided into two steps: analysis and decomposition. The analysis step primarily preprocesses the original matrix and provides an appropriate data structure for the decomposition step. The decomposition step mainly involves establishing an elimination tree. Then, the inverse matrix of the preconditioning matrix $C_{11}^H C_{11}$ is

$$(C_{11}^H C_{11})^{-1} \approx \mathbf{P}_r^T \mathbf{D}_r^T (\mathbf{L}\mathbf{L}^H)^{-1} \mathbf{D}_r \mathbf{P}_r^T \quad (3)$$

Where $\mathbf{P}_r$ is a reordered matrix. $\mathbf{D}_r$ is a diagonal scaling matrix, and there is an approximate processing $\mathbf{L}\mathbf{L}^H \approx \mathbf{D}_r \mathbf{P}_r (C_{11}^H C_{11}) \mathbf{P}_r^T \mathbf{D}_r^T$. It is particularly worth noting that this MFBIC preconditioning method is also applicable to non-symmetric matrices. It only requires replacing the incomplete Cholesky decomposition method with the incomplete LU decomposition method (ILU) for computation.

In the IB-BGMRES-DR block iterative algorithm, to reduce memory consumption from matrix-matrix multiplication, considering that the right-hand side is a matrix rather than a column vector and the number of right-hand sides is far fewer than matrix dimensions, hence we use a for loop to separately preprocess each right-hand side vector when using the MFBIC preconditioner. As the preconditioning only needs to be performed once before the iteration of the IB-BGMRES-DR block iterative algorithm, this method is effective. Subsequently, when the actual iterative begins, these preprocessed vectors are combined into a single matrix. By applying the aforementioned two preconditioner, Eq. (1) is transformed into the following equation that is easier to solve.

$$\mathcal{M}^{-1} \mathbf{A} \mathbf{X} = \mathcal{M}^{-1} \mathbf{B} \quad (4)$$

Following the IB-BGMRES-DR algorithm, the block matrix of all approximate solutions can be output at once.

III. NUMERICAL ANALYSIS

In this section, numerical test is X-51 hypersonic vehicle. The calculation frequency is 100 MHz. This example primarily analyzes the monostatic RCS values within the range of $\varphi = 0$ and $\theta = [-160^\circ, -120^\circ]$ with interval of 1° , corresponding to 41 right-hand side vectors. This model is discretized into 15,066 tetrahedrons, resulting in a matrix dimension of 98,732. The error threshold for the simulation is set to $\varepsilon = 10^{-8}$. The number of matrix-vector multiplications is defined as *mvps*. To verify the correctness of IB-BGMRES-DR algorithm with this proposed preconditioning method, we use GMRES and HFSS software for comparisons. As

illustrated in Figure 1, the IB-BGMRES-DR block iterative algorithm proposed in this paper exhibits a consistent RCS variation trend. Besides, we give the surface electric field distribution plot of this model, which demonstrates that the implementation of preconditioning technique is correct in block iterative algorithm. In Table I, we further provide a comparison of computational performance. From these data, we can observe that the proposed IB-BGMRES-DR requires fewer matrix-vector multiplications and has a shorter overall simulation time. In the future, we will further study the restart mechanism of this algorithm and its parallel technique to achieve better computational performance.

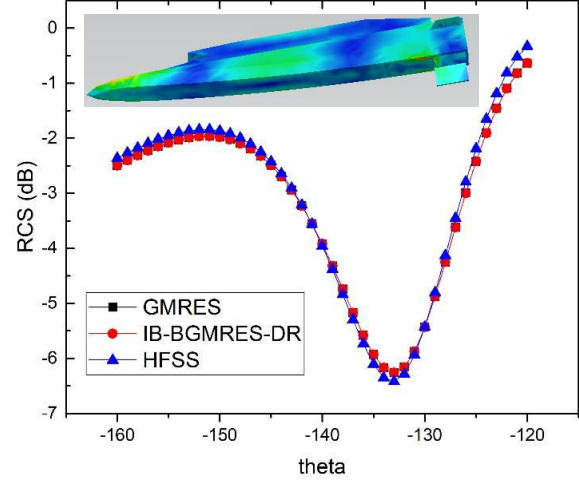


Figure 1. The monostatic radar cross section (RCS) for the X-51 model.

TABLE I. COMPARISON OF COMPUTATIONAL PERFORMANCE BETWEEN GMRES AND IB-BGMRES-DR METHODS.

methods	mvps	time (s)	memory (M)
GMRES	164	13	324
IB-BGMRES-DR	47	8	389

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