

Arbitrary Transverse Magnetic Multipole Radiation for Spherical Electric Current Density via Exact Maxwell Solution

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Abstract—Recent analyses have included the generation of new designs for transverse electric (TE) antennas for which the electric current flows within a spherical surface. These calculations give exact solutions of Maxwell’s equations for all spatial positions. In the present investigation, transverse magnetic (TM) solutions of Maxwell’s equations are computed corresponding to general non-uniform electric currents on a spherical surface by using vector spherical harmonic (VSH) functions. Thus, this investigation facilitates the calculation of radiative spherical antenna properties for higher frequencies, as is probable for general non-uniform electric current profiles within the spherical surface.

I. INTRODUCTION

Many past studies of magnetic antennas examined electric current loops [1]–[5]. Recent studies [6]–[12] have included the derivation of exact Maxwell solutions and radiation properties for antennas in which the electric current density flows within a spherical surface. The investigations in Refs. [6], [8] applied techniques used previously for thin current loops, as developed in Refs. [1]–[5]. A generalization for higher order modes is performed by applying the vector spherical harmonic (VSH) functions Berrera et al. [13] and Carrascal et al. [14]. A recent investigation [11] included the calculation of exact transverse electric (TE) Maxwell solutions for general electric current profiles within a spherical surface by using these VSHs. In the present analysis, exact Maxwell solutions are calculated for transverse magnetic (TM) fields due to general non-uniform electric currents on a spherical surface.

II. VECTOR SPHERICAL HARMONICS

The present analysis uses standard spherical coordinates $\mathbf{r} = \{r, \theta, \phi\}$, with the coordinate origin set equal to the center of the spherical current distribution. This investigation assumes spatially uniform and temporally constant values for the electric permittivity ε and magnetic permeability μ , with different values possible outside versus inside of the spherical current. The scalar spherical harmonics (SSH) $Y_{\ell,m}(\theta, \phi)$ are (e.g., Ref. [15], Eqns. (3.53) and (3.54)):

$$Y_{\ell,m}(\theta, \phi) = \sqrt{\frac{\{2\ell+1\}}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^m(\cos(\theta)) e^{im\phi}, \quad (1)$$

$$Y_{\ell,-m}(\theta, \phi) = (-1)^m Y_{\ell,m}(\theta, \phi),$$

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in terms of the associated Legendre functions $P_{\ell}^m(u)$. The present analysis uses the VSHs of Berrera et al. [13], yielding the following form general 3-D vector field $\mathbf{H}(\mathbf{r})$ (e.g., the magnetic field):

$$\mathbf{H}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \{ H_{\ell,m}^{\{r\}}(r) \mathbf{Y}_{\ell,m}(\theta, \phi) + H_{\ell,m}^{\{1\}}(r) \mathbf{\Psi}_{\ell,m}(\mathbf{r}) + H_{\ell,m}^{\{2\}}(r) \mathbf{\Phi}_{\ell,m}(\mathbf{r}) \}. \quad (2)$$

$$\begin{aligned} \mathbf{Y}_{\ell,m}(\theta, \phi) &= Y_{\ell,m}(\theta, \phi) \hat{\mathbf{r}}(\theta, \phi) \\ \mathbf{\Psi}_{\ell,m}(\mathbf{r}) &= r \nabla Y_{\ell,m}(\theta, \phi), \\ \mathbf{\Phi}_{\ell,m}(\mathbf{r}) &= \mathbf{r} \times \nabla Y_{\ell,m}(\theta, \phi). \end{aligned} \quad (3)$$

III. TRANSVERSE MAGNETIC MODES

In the present investigation, exact TM “electric” radiation mode solutions are calculated for the case in which the source electric currents and electric charge densities on constrained to lie within a spherical surface of radius a . The zero-divergence property of magnetic induction together with VSH properties yield the following 3-D vector form of the magnetic field:

$$\mathbf{H}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} H_{\ell,m}^{\{2\}}(r) \mathbf{\Phi}_{\ell,m}(\mathbf{r}). \quad (4)$$

To solve Ampere’s law for the case of TM modes with $\mathbf{J}(\mathbf{r}) = \mathbf{0}$ for $r < a$ and $r > a$, examine Faraday’s law for TE modes (Ref. [12], Eqs. (16)-(20)). Examination of Ampere’s law for TM fields with Faraday’s law for TE fields implies the following replacements can be applied Faraday’s law for TE modes:

$$\mathbf{E}(\mathbf{r}) \longleftrightarrow \mathbf{H}(\mathbf{r}), \quad \varepsilon \longleftrightarrow \mu, \quad i \longleftrightarrow -i \quad (5)$$

to calculate TM solutions of Ampere’s law. Thus, the TM electric field has the form:

$$\mathbf{E}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \{ E_{\ell,m}^{\{r\}}(r) \mathbf{Y}_{\ell,m}(\theta, \phi) + E_{\ell,m}^{\{1\}}(r) \mathbf{\Psi}_{\ell,m}(\mathbf{r}) \}, \quad (6)$$

$$\begin{aligned} E_{\ell,m}^{\{r\}}(r) &= i \frac{\ell\{\ell+1\}}{\omega\varepsilon} \frac{1}{r} H_{\ell,m}^{\{2\}}(r) \\ E_{\ell,m}^{\{1\}}(r) &= i \frac{1}{\omega\varepsilon} \frac{1}{r} \frac{d}{dr} \{ r H_{\ell,m}^{\{2\}}(r) \}. \end{aligned} \quad (7)$$

Next, examine Coulomb’s law for spatial regions both exterior and interior to the spherical current distribution wherein the electric charge density is zero. The TM solution

of Coulomb's law for $\rho(\mathbf{r}) = 0$ for $r < a$ and $r > a$ is calculated by using the zero divergence of the magnetic field, as developed in Ref. [11], Eqs. (21)-(23). Thus, the replacements of (5) are applied to yield the divergence of TM electric fields:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \frac{dE_{\ell,m}^{\{r\}}}{dr} + \frac{2}{r} E_{\ell,m}^{\{r\}}(r) - \ell\{\ell+1\} \frac{1}{r} E_{\ell,m}^{\{1\}}(r) \right\} Y_{\ell,m}(\theta, \phi). \quad (8)$$

Application of (7) in (8) is consistent with $\nabla \cdot \mathbf{E}(\mathbf{r}) = 0$ for $r < a$ and $r > a$.

To solve Faraday's law for TM fields, apply Ampere's law as examined in the prior investigation involving TE fields (Ref. [11], Eqs. (16)-(18)) by using (5) with $\mathbf{J}(\mathbf{r}) = \mathbf{0}$ for $r < a$ and $r > a$. Therefore, Faraday's law for TM fields gives:

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \frac{1}{r} \frac{d}{dr} \left\{ r E_{\ell,m}^{\{1\}}(r) \right\} - \frac{1}{r} E_{\ell,m}^{\{r\}}(r) \right\} \Phi_{\ell,m}(\mathbf{r}) \\ &= -i\omega\mu \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} H_{\ell,m}^{\{2\}}(r) \Phi_{\ell,m}(\mathbf{r}), \end{aligned} \quad (9)$$

which can be expressed as the following ordinary differential equation by applying (5) in the prior investigation involving TE fields (Ref. [11], Eq. (26)):

$$\frac{1}{r} \frac{d}{dr} \left\{ r E_{\ell,m}^{\{1\}}(r) \right\} - \frac{1}{r} E_{\ell,m}^{\{r\}}(r) = -i\omega\mu H_{\ell,m}^{\{2\}}(r). \quad (10)$$

Then, the use of (7) within (10) for $r < a$ and $r > a$ gives:

$$r^2 \frac{d^2 H_{\ell,m}^{\{2\}}}{dr^2} + 2r \frac{dH_{\ell,m}^{\{2\}}}{dr} + \{k^2 r^2 - \ell\{\ell+1\}\} H_{\ell,m}^{\{2\}}(r) = 0, \quad (11)$$

with wavenumber $k = 1/\lambda$ and wavelength λ .

Analysis of (11) shows that it corresponds to the spherical Bessel equation for $H_{\ell,m}^{\{2\}}(r)$. In particular, the TM equation (11) for $H_{\ell,m}^{\{2\}}(r)$ is the same as the TE equation for $E_{\ell,m}^{\{2\}}(r)$ in the prior investigation involving TE fields (Ref. [11], Eq. (33)). Therefore, the use of (11) with the requirements of finite field energy inside of the current sphere and outwardly propagating electromagnetic waves exterior to the sphere yields the following form for $H_{\ell,m}^{\{2\}}(r)$:

$$H_{\ell,m}^{\{2\}}(r) = \alpha_{\ell,m} g_{\ell}(r), \quad (12)$$

with $\alpha_{\ell,m}$ giving constants. In addition, $g_{\ell}(r)$ has the form:

$$g_{\ell}(r) \equiv \begin{cases} \xi_{\ell,m} j_{\ell}(kr), & \text{for } r < a \\ \psi_{\ell,m} h_{\ell}^{(2)}(kr), & \text{for } r > a \end{cases}, \quad (13)$$

such that integral form of Ampere's law determines the relationship between the constants $\xi_{\ell,m}$ and $\psi_{\ell,m}$. Similarly, the TM electric fields are determined by (7) with:

$$E_{\ell,m}^{\{r\}}(r) = i\alpha_{\ell,m} \frac{\ell\{\ell+1\}}{\omega\epsilon} \frac{1}{r} g_{\ell}(r), \quad (14)$$

$$E_{\ell,m}^{\{1\}}(r) = i\alpha_{\ell,m} \frac{1}{\omega\epsilon} \frac{1}{r} \frac{d}{dr} \left\{ r g_{\ell}(r) \right\}. \quad (15)$$

IV. CONCLUSIONS

The present study has investigated the form of electromagnetic radiation emanated by a spherical electric current shell for TM "electric" fields. This analysis has yielded closed-form expressions for the electromagnetic fields corresponding for the various radiation modes, wherein near-field contributions are retained. Thus, this investigation offers to provide more realistic models corresponding to electric current profiles that are arbitrary in permitting the inclusion of both spherical polar and spherical azimuthal components, which can be non-uniform as well. Therefore, this study offers to enable accurate antenna modeling at higher frequencies such that higher order electric current characteristics can be expected to be more significant.

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