Completeness of the Characteristic Mode Expansion

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Characteristic mode theory was originally developed as a methodology to diagonalize the scattering matrix for arbitrarily shaped objects (Garbacz 1968, Harrington and Mautz 1971), thereby generalizing some of the fundamental properties from the spherical wave expansion and Mie series for spheres. Here, we revisit and extend the original derivations using a spherical mode decomposition $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathrm{H}}\mathbf{S}$ of the radiation matrix (Tayli *et al.* 2018) together with the observations that the excitation of an incident wave expanded in regular spherical waves (Kristensson 2016) can be written as $\mathbf{S}^{\mathrm{H}}\mathbf{a}$ (expansion coefficients in $\mathbf{a}$) and the expansion of the radiated field in outgoing spherical waves as $\mathbf{f} = \mathbf{S}\mathbf{I}$ for a current $\mathbf{I}$.

The new derivation shows that the characteristic mode decomposition diagonalizes the transition matrix using simple algebraic manipulations and thereby significantly reduces the classical derivations. The generalized eigenvalue problem for characteristic modes is first transformed to an eigenvalue problem for outgoing spherical waves (Tayli *et al.* 2018)

$$\mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}_{\mathrm{r}}\mathbf{I}_n \Rightarrow \mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{f}_n = \lambda_n^{-1}\mathbf{f}_n \quad \text{with } \mathbf{I}_n = \lambda_n \mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{f}_n, \quad \mathbf{f}_n = \mathbf{S}\mathbf{I}_n.$$

Expanding the incident field in regular spherical waves $\mathbf{V} = \mathbf{S}^{\mathrm{H}}\mathbf{a}$ and substitution into the MoM formula multiplied with $\mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}}$ produces

$$\mathbf{Z}\mathbf{I} = (\mathbf{R}_{\mathrm{r}} + \mathrm{j}\mathbf{X})\mathbf{I} = (\mathbf{S}^{\mathrm{H}}\mathbf{S} + \mathrm{j}\mathbf{X})\mathbf{I} = \mathbf{V} \Rightarrow (\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}} + \mathrm{j}\mathbf{1})\mathbf{f} = \mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{a}.$$

Here it is seen that the transition matrix $\mathbf{T}$ mapping regular spherical waves to outgoing spherical waves, $\mathbf{f} = \mathbf{T}\mathbf{a}$, is diagonalized by characteristic modes.

Completeness of the characteristic mode expansion is discussed from the perspective of scattered (or radiated) field (expansion coefficients $\mathbf{f} = \mathbf{S}\mathbf{I}$) and current density ($\mathbf{I}$). We show that expansion of the scattered field is complete. Completeness of the current density is more involved and here we show that characteristic modes are compatible with excitation expressed in terms of regular spherical waves only, *i.e.*,

$$\mathbf{Z}\mathbf{I}_n = (\mathbf{S}^{\mathrm{H}}\mathbf{S} + \mathrm{j}\mathbf{X})\lambda_n\mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{f}_n = \mathbf{S}^{\mathrm{H}}(\mathbf{S}\mathbf{X}^{-1}\mathbf{S}^{\mathrm{H}} + \mathrm{j}\mathbf{1})\mathbf{f}_n\lambda_n = \mathbf{S}^{\mathrm{H}}\mathbf{f}_n(1 + \mathrm{j}\lambda_n) = \mathbf{V}_n$$

Therefore, characteristic modes can be used to expand current densities corresponding to incident waves which can be expanded in regular spherical waves ($\mathbf{a}_n = (1 + \mathrm{j}\lambda_n)\mathbf{f}_n$) such as plane waves or near fields with sources supported outside a sphere circumscribing the object. An important case which does not satisfy this requirement is that of localized on-structure feeding, e.g., delta gap excitation, explicitly demonstrating that driven antenna currents may not be representable by characteristic modes in the strictest sense. The results are illustrated with numerical examples for PEC and lossless dielectric objects.