Half-Way Duality and the Fractional Curl Operator

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***Abstract*—Half-dual formulations of the electromagnetic field equations could provide efficient computational shortcuts for scattering problems, in the same way that full dual representations do. Such a formulation is presented herein, based on a general formal derivation of the half curl. For the case of a plane wave, the half-dual formulation translates to a 45^0 tilt of both fields and sources relative to conventional electric and magnetic fields and sources, in analogy to the 90^0 tilt that coincides with the full dual.**

I. Introduction

Fractional derivatives have found applications in many branches of science, including electromagnetics [1], [2] and quantum mechanics [3]. A number of definitions allow for the generalization an integer order derivative to fractional order. The classic Riemann-Liouville definition and other definitions seem to have issues with singular functions such as the step function and with distributions [4]. Another way for finding the half-order derivatives has been presented in [5], where the operator equation

$$(\nabla\times)^{\frac{1}{2}}\cdot(\nabla\times)^{\frac{1}{2}}=\nabla\times \tag{1}$$

was formally solved for the half-curl operator $(\nabla\times)^{\frac{1}{2}}$. We use this formulation to produce a "half-way" dual for Maxwell's curl equations. As noted in [2], the full dual coincides with a 90^0 tilt of plane wave fields, and the half-way dual is essentially a 45^0 tilt. We show that auxiliary fields, denoted $\mathbf{K}$ and $\mathbf{K}$R, can replace $\mathbf{E}$ and $\mathbf{H}$ as their half-dual, 45^0 tilted version. the auxiliary fields are fo the same dimension, which can be seen as "half-way" vetwenn V/m and A/m. A set of new sources, also of the same dimension are found to be are tilted with respect to the physical electric and magnetic currents. This formulation opens the door for obtaining a straightforward solution for a problem whose half-dual version is known.

II. Half-Dual Maxwell's Equations

The half order curl operator has been evaluated in two dimensions in [5] and used to derive equations for a pair of auxiliary fields, denoted $\mathbf{K}$ and $\mathbf{R}$, that can be used as alternatives to the conventional frequency domain $\mathbf{E}$ and $\mathbf{H}$ fields in the course of EM computations. A four-field set of half-order state-space equations was then formulated [5]:

$$(\nabla\times)^{\frac{1}{2}}\mathbf{E}=-(j\omega\mu_1)^{\frac{1}{2}}\mathbf{K}-(\nabla\times)^{\frac{1}{2}}\mathbf{M}_2 \tag{2a}$$
$$(\nabla\times)^{\frac{1}{2}}\mathbf{H}=(j\omega\epsilon_1)^{\frac{1}{2}}\mathbf{R}+(\nabla\times)^{\frac{1}{2}}\mathbf{J}_1 \tag{2b}$$
$$(\nabla\times)^{\frac{1}{2}}\mathbf{K}=(j\omega\mu_2)^{\frac{1}{2}}\mathbf{H}+\mathbf{J}_2 \tag{2c}$$
$$(\nabla\times)^{\frac{1}{2}}\mathbf{R}=(j\omega\epsilon_2)^{\frac{1}{2}}\mathbf{E}+\mathbf{M}_1, \tag{2d}$$

from which we formulate the following set of augmented Maxwell's equations with the new source terms $\mathbf{P}$ and $\mathbf{Q}$:

$$\nabla\times\mathbf{E}=-\jmath\omega\mu\mathbf{H}-\mathbf{M} \tag{3a}$$
$$\nabla\times\mathbf{H}=\jmath\omega\epsilon\mathbf{E}+\mathbf{J} \tag{3b}$$
$$\nabla\times\mathbf{K}=\jmath\omega(\mu\epsilon)^{\frac{1}{2}}\mathbf{R}+\mathbf{P} \tag{3c}$$
$$\nabla\times\mathbf{R}=-\jmath\omega(\epsilon\mu)^{\frac{1}{2}}\mathbf{K}-\mathbf{Q}. \tag{3d}$$

having set

$$(\epsilon_1\epsilon_2)^{\frac{1}{2}}=\epsilon,\quad(\mu_1\mu_2)^{\frac{1}{2}}=\mu. \tag{4}$$

The source terms in (3) are related to those in (2) via

$$(j\omega\mu_1)^{\frac{1}{2}}\mathbf{J}_2+\nabla\times\mathbf{M}_2=\mathbf{M} \tag{5a}$$
$$(j\omega\epsilon_1)^{\frac{1}{2}}\mathbf{M}_1+\nabla\times\mathbf{J}_1=\mathbf{J}. \tag{5b}$$
$$(j\omega\mu)^{\frac{1}{2}}(\nabla\times)^{\frac{1}{2}}\mathbf{J}_1+(\nabla\times)^{\frac{1}{2}}\mathbf{J}_2=\mathbf{P} \tag{5c}$$
$$-(j\omega\epsilon)^{\frac{1}{2}}(\nabla\times)^{\frac{1}{2}}\mathbf{M}_2+(\nabla\times)^{\frac{1}{2}}\mathbf{M}_1=-\mathbf{Q}. \tag{5d}$$

The two curl equations (3a)-(3b) are the dual of each other. As pointed out in [2], when it comes to the representation of a plane wave, duality and a 90^0 tilt are the same. In an analogous way, the set (3a)-(3b) can be considered "half dual" to (3c)-(3d). The use of the conventional or dual pair would now be driven by the problem at hand.

A. Example: Homogeneous Plane Wave

A treatment for a general order partial derivative is given in [2]. Revisiting this example, we look at the following plane wave

$$\mathbf{E}(x)=\hat{\mathbf{y}}E_0e^{-jkx} \tag{6a}$$
$$\mathbf{H}(x)=\hat{\mathbf{z}}\frac{E_0}{\eta}e^{-jkx}. \tag{6b}$$

Invoke (2):

$$\mathbf{K}=-(j\omega\mu)^{-\frac{1}{2}}(\nabla\times)^{\frac{1}{2}}\mathbf{E} \tag{7a}$$
$$\mathbf{R}=(j\omega\epsilon)^{-\frac{1}{2}}(\nabla\times)^{\frac{1}{2}}\mathbf{H} \tag{7b}$$

In order to evaluate $\mathbf{K}$ and $\mathbf{K}$, we reduce the formal expression for the half-curl that was derived in [5], i.e.,

$$(\nabla\times)^{\frac{1}{2}} = \frac{j\triangledown^{\frac{1}{2}}}{\sqrt{2}\nabla^2}\cdot \begin{pmatrix} \mathcal{D}_y^2+\mathcal{D}_z^2 & -\mathcal{D}_x\mathcal{D}_y+\triangledown\mathcal{D}_z & -\mathcal{D}_z\mathcal{D}_x-\triangledown\mathcal{D}_y \\ -\mathcal{D}_x\mathcal{D}_y-\triangledown\mathcal{D}_z & \mathcal{D}_x^2+\mathcal{D}_z^2 & -\mathcal{D}_y\mathcal{D}_z+\triangledown\mathcal{D}_x \\ -\mathcal{D}_z\mathcal{D}_x+\triangledown\mathcal{D}_y & -\mathcal{D}_y\mathcal{D}_z-\triangledown\mathcal{D}_x & \mathcal{D}_x^2+\mathcal{D}_y^2 \end{pmatrix} \quad (8)$$

(where $\triangledown = \sqrt{\nabla^2}$) to our case $\mathcal{D}_y = \mathcal{D}_z = 0$, where $\triangledown = \mathcal{D}_x$:

$$(\nabla_t\times)^{\frac{1}{2}} = \frac{j}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\mathcal{D}_x^{\frac{1}{2}}, \quad (9)$$

For (9), $\mathcal{D}_x^{\frac{1}{2}} = (-jk)^{\frac{1}{2}}$.

$$\mathbf{K} = \left(\frac{k}{2\omega\mu}\right)^{\frac{1}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ E_0e^{-jkx} \\ 0 \end{pmatrix} = \frac{E_0e^{-jkx}}{(2\eta)^{\frac{1}{2}}}(\hat{\mathbf{y}}-\hat{\mathbf{z}}) \quad (10a)$$

$$\mathbf{R} = -\left(\frac{k}{2\omega\epsilon}\right)^{\frac{1}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \frac{E_0}{\eta}e^{-jkx} \end{pmatrix} = \frac{E_0e^{-jkx}}{(2\eta)^{\frac{1}{2}}}(\hat{\mathbf{y}}+\hat{\mathbf{z}}), \quad (10b)$$

While a full diality coincides with a 90^0 rotation along the axis of propagation, this half way duality is in essence a 45^0 tilt. Also, in a full duality, Volts and Ampères are interchanged. Half way through this transition, both $\mathbf{K}$ and $\mathbf{K}$ are of the same dimension. Also,

$$\mathbf{K}\cdot\mathbf{R} = 0; \quad \frac{1}{2}\mathbf{K}\times\mathbf{R}^\star = \frac{|E_0|^2}{2\eta}\hat{\mathbf{x}}, \quad (11)$$

I.e., $\mathbf{K}$ and $\mathbf{R}$ preserve the plane wave property of being orthogonal to each other and to the direction of propagation, and also preserve the value of the Poynting vector.

B. Plane Wave over Half Space

Equations (6) are now modified into the half-space case:

$$\mathbf{E}(x) = \hat{\mathbf{y}}E_0e^{-jk|x|} \quad (12a)$$

$$\mathbf{H}(x) = \hat{\mathbf{z}}\frac{E_0}{\eta}e^{-jk|x|}\mathrm{sgn}(x). \quad (12b)$$

The results for the half-space $x > 0$ remain the same as in Sec. II-A. However, in this case, the wave is generated by the following source at $x = 0$:

$$\mathbf{J} = -\hat{\mathbf{y}}\frac{2E_0}{\eta}\delta(x). \quad (13)$$

The full dual of (12) is

$$\mathbf{H}(x) = \hat{\mathbf{y}}\frac{E_0}{\eta}e^{-jk|x|} \quad (14a)$$

$$\mathbf{E}(x) = -\hat{\mathbf{z}}E_0e^{-jk|x|}\mathrm{sgn}(x). \quad (14b)$$

with the sources

$$\mathbf{M} = -\hat{\mathbf{y}}2E_0\delta(x). \quad (15)$$

One half dual of (12) is

$$\mathbf{K} = \frac{E_0e^{-jk|x|}}{(2\eta)^{\frac{1}{2}}}(\hat{\mathbf{y}}-\hat{\mathbf{z}}) \quad (16a)$$

$$\mathbf{R} = \frac{E_0e^{-jk|x|}}{(2\eta)^{\frac{1}{2}}}(\hat{\mathbf{y}}+\hat{\mathbf{z}})\mathrm{sgn}(x) \quad (16b)$$

with the sources, as extracted from(3),

$$\mathbf{P} = \nabla\times\mathbf{K} - \jmath k\mathbf{R} = -\jmath k\frac{E_0}{(2\eta)^{\frac{1}{2}}}(-\hat{\mathbf{y}}-\hat{\mathbf{z}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})\mathrm{sgn}(x) = 0 \quad (17a)$$

$$\mathbf{Q} = -\jmath k\mathbf{K} - \nabla\times\mathbf{R} = -\frac{2E_0}{(2\eta)^{\frac{1}{2}}}(\hat{\mathbf{y}}-\hat{\mathbf{z}})\delta(x) = \mathbf{Q}_s\delta(x). \quad (17b)$$

We see that $\mathbf{Q}_s = -\frac{2E_0}{(2\eta)^{\frac{1}{2}}}(\hat{\mathbf{y}}-\hat{\mathbf{z}}) = \hat{\mathbf{x}}\times\mathbf{R}|_{x=0}$ is a surface source at $x = 0$.

III. Conclusions

References

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