6-D Integrals for Numerical Evaluation by Double Application of the Divergence Theorem

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***Abstract*—In this paper we propose a scheme to treat, as a whole, the 6-D reaction integrals appearing in the Method of Moments. The divergence theorem is twice applied directly in the physical space domain. The resulting 6-D volume integrals are expressed as two radial integrals plus four linear integrals over the source and observation domain boundaries. The method is numerically validated for static and dynamic kernels arising in the Electric Field Integral Equation (EFIE), i.e., for kernels with 1/R singularities, and linear basis functions.**

I. Introduction

Volume integral equation (VIE) techniques based on the method of moments (MoM) are especially useful in cases involving inhomogeneous materials either in the direct scattering problem or in the inverse scattering problem, as in microwave imaging techniques. However, the rigorous solution of radiation and scattering problems using VIEs requires the accurate and efficient numerical evaluation of double volume reaction integrals. Recently, powerful numerical methods for handling the entire integration of surface integrals on test and observation element pairs in moment method solutions have been demonstrated in several papers, e.g. see [1]–[3]. Their approach is based on an interchange of integration orders to first perform radial integrations for both source and observation point integrals. However, the approach was limited to surface elements. In [4] a method allowing an analytical conversion of expressions for matrix elements of the tensor and vector Green functions from 6-D volumetric to 4-D surface integrals with nonsingular integrands was presented.

In this paper we extend the applicability of the method of [2], [3] to the evaluation of double volumetric integrals on source and test domains of the 6-D reaction integrals, which also produces radial source and test integrals. The divergence theorem is applied directly in the physical domains for both the source and observation point integrals, and the resulting radial integrals are well-behaved for polygonal domains. In this sense, the scheme is quite general, i.e., is not limited to well-shaped elements nor to ad-hoc treatments of self-, edge-, or vertex-adjacent geometries. Furthermore, we introduce a parameterization that will allow us to use the transformations of variables implemented in [2], [3] and that have proven their effectiveness in accelerating the accuracy of the integrals.

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II. Formulation

We consider first the inner volume integral

$$\int_{V'} F(\mathbf{r},\mathbf{r}')dV'. \tag{1}$$

If we can find a vector function $\mathbf{H}(\mathbf{r},\mathbf{r}')$ such that $\nabla'\cdot\mathbf{H} = F(\mathbf{r},\mathbf{r}')$, then by the divergence theorem (1) can be written as

$$\int_{V'} F(\mathbf{r},\mathbf{r}')dV' = \oint_{S'} \mathbf{H}(\mathbf{r},\mathbf{r}'_{S'})\,\cdot\,\hat{\mathbf{n}}'dS', \tag{2}$$

where the volume integral over a tetrahedral element, such as shown in Fig. 1.a, is reduced to a surface integral over its faces. $\mathbf{r}'_{S'}$ is a point on the boundary S' of V', $\hat{\mathbf{n}}'$ is the external normal to the boundary surface of the tetrahedral volume, and

$$\mathbf{H}(\mathbf{r},\mathbf{r}'_{S'}) \;=\; \frac{\hat{\mathbf{R}}'}{R^2}\int_0^R F(\mathbf{r},\mathbf{r}')\,{R'}^2dR', \tag{3}$$

where $\mathbf{r}' = \mathbf{r} + R'\hat{\mathbf{R}}'$, $\hat{\mathbf{R}}' = (\mathbf{r}'-\mathbf{r})/R'$, $0 \le R' \le\; R \equiv |\,\mathbf{r}-\mathbf{r}'_{S'}|$, $R' = |\mathbf{r}'-\mathbf{r}|$ as seen in Fig. 1.a. Using (3) we can write (2) as

$$\int_{V'} F(\mathbf{r},\mathbf{r}')dV' = \oint_{S'}\left(\frac{\hat{\mathbf{n}}'\cdot\hat{\mathbf{R}}'}{R^2}\int_0^R F(\mathbf{r},\mathbf{r}')\,{R'}^2dR'\right)dS'. \tag{4}$$

Next, we consider an additional testing integration over the planar observer domain S in order to evaluate the following double surface integral:

$$\begin{aligned}&\int_V\int_{V'} F(\mathbf{r},\mathbf{r}')dV'dV \;= \\ &\oint_{S'}\hat{\mathbf{n}}'\cdot\left[\int_V\left(\frac{\hat{\mathbf{R}}'}{R^2}\int_0^R F(\mathbf{r},\mathbf{r}')\,{R'}^2dR'\right)dV\right]dS',\end{aligned} \tag{5}$$

where the interchange of integration order is permitted by the independence of the source and observation coordinates and their associated domains. Again applying the divergence theorem to the volumetric integral in (5) and rearranging the order of the resulting integrals, we obtain

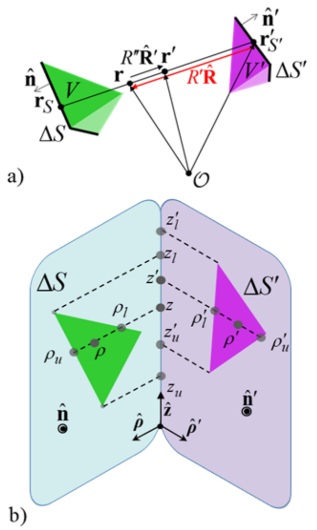

Fig. 1. Geometry definitions: a) The orientation of a pair of tetrahedral elements in space, b) Geometry definitions for integrating over a line segment pair.

$$\oint_{S}\oint_{S'}\frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{R}})(\hat{\mathbf{n}}'\cdot\hat{\mathbf{R}}')}{{R_{SS'}}^{2}}\int_{0}^{R_{SS'}}\int_{0}^{R}F(\mathbf{r},\mathbf{r}'){R'}^{2}dR'dRdS'dS, \quad (6)$$

where S is the boundary of V, $\hat{\mathbf{n}}$ is the outward pointing normal to S, $\mathbf{r} = \mathbf{r}'_{S'} + R\hat{\mathbf{R}}$, $\hat{\mathbf{R}} = -\hat{\mathbf{R}}' = (\mathbf{r}_S - \mathbf{r}_{\mathbf{S}'})/R_{SS'}$, $0 \leq R \leq R_{SS'}$, $R_{SS'} = |\mathbf{r}_S - \mathbf{r}'_{S'}|$, and $\mathbf{r}_S$ is a point on S. The radial integrals can be performed in closed form for both the dynamic and static forms of the free space Greens function $G\left(|\mathbf{r}-\mathbf{r}'|^{-1}\right)$ and grad $\left(|\mathbf{r}-\mathbf{r}'|^{-1}\right)$, with or without polynomial vector bases. To analyzed the two outer surface integrals we apply variable transformation for both surfaces. If we consider

$$\mathcal{F}(\mathbf{r},\mathbf{r}') = \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{R}})(\hat{\mathbf{n}}'\cdot\hat{\mathbf{R}}')}{{R_{SS'}}^{2}}\int_{0}^{R_{SS'}}\int_{0}^{R}F(\mathbf{r},\mathbf{r}'){R'}^{2}dR'dR\,, \quad (7)$$

the integral (6) can be written as

$$\int_{\Delta S}\int_{\Delta S'}\mathcal{F}(\mathbf{r},\mathbf{r}')dS'dS = \int_{z}\int_{\rho}\int_{z'}\int_{\rho'}\mathcal{F}(z,\rho,z',\rho')d\rho'dz'd\rho dz, \quad (8)$$

where the surfaces ΔS and $\Delta S'$ have been parameterized using the intersection line with orientation ($\hat{\mathbf{z}}$) between these surfaces and the vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\rho}}'$ orthogonal to the intersection line and the normal to the plane containing the surface ΔS and $\Delta S'$ respectively (Fig. 1.b). This parameterization will let us use some of the variable transformations developed in [2].

III. Preliminary Numerical Results

To show the accuracy of the 6-D reaction integral reported in (6), we examine the scalar potential in the Method of Moments discretization of the Electric Field Integral Equation (EFIE). We consider a pair of source and test tetrahedra with a common edge, as shown in Fig. 2 (inset). An exact result is used for the radial integrals in Fig. 2, which then compares the standard Gauss-triangle (GT) quadrature scheme [5] to a reference result obtained using the GT scheme with the highest number of points we have available for this scheme (166 points). In Fig. 2, the convergence of the different sets of interaction faces is separately investigated. Fig. 2 shows the number of correct significant digits obtained increasing the number of surface sample points for the boundary integrals. The sets of non-touching faces show the best convergence, as expected, approaching machine precision accuracy with about 16 points per linear dimension.

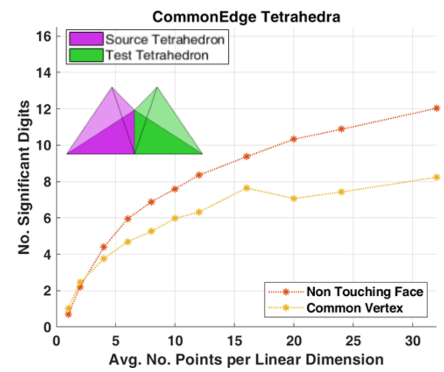

Fig. 2. Near-field convergence of surface integrals. Inset: Orientation of a pair of tetrahedral elements in space.

IV. Conclusion

The proposed scheme is based on two applications of the divergence theorem with an appropriate integration reordering and a variable transformation. The 6-D integrals are expressed as two radial integrals plus four linear integrals over source and observation domain boundaries. The method is numerically validated for static and dynamic kernels arising in the EFIE and similar formulations. The next step in this research activity will be to examine the possibility of using other transformations to further smooth the resulting integrands and hence accelerate their convergence.

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