A Discontinuous Galerkin Method for the Solution of Two Dimensional Axisymmetric Radiative Transfer Problem

H. Wang, R. Abedi,
Department of Mechanical Aerospace and Biomedical Engineering
University of Tennessee Space Institute
Manchester, TN 37388-9700, USA
hwang@utsi.edu,rabedi@utsi.edu

S. Mudaliar
Sensors Directorate
Air Force Research Laboratory, Wright-Patterson AFB
Dayton, OH 45433-7318, USA
saba.mudaliar@us.af.mil

***Abstract*—The radiative transfer equation (RTE) for an axisymmetric problem involving scattering, absorption and radiation is solved using the discontinuous Galerkin (DG) finite element method (FEM). Both space and angle directions are discretized by the DG method. The problem is formulated for nonzero phase function. The method is validated against exact solutions. A few benchmark problems are conducted. The performance of the method is also studied for the solution of problems with discontinuous solution.**

I. INTRODUCTION

The radiative transfer equation (RTE) describes the propagation of radiation in the form of electromagnetic waves through a medium affected by absorption, emission, and scattering processes. Over the past 50 years, several techniques for solving the multi-dimensional RTE have been introduced. These include, but are certainly not limited to, Monte Carlo methods, discrete-ordinate methods, spherical harmonics methods, spectral methods, finite difference methods, and finite element methods. Methods involving discrete ordinates and spherical harmonics have received particular attention in the literature. In spite of their popularity, the discrete ordinate method and spherical harmonics method are not the only methods used to solve the RTE. Discontinuous Galerkin (DG) methods relax the continuity constraint of continuous finite element methods (FEM), where jump solution between elements not only is enforced weakly but also is based on wave propagation direction. Hybrid methods with DG discretization in space are presented for discrete ordinate discretization in angle direction[**?**]. In our previous work, the RTE for a plane-parallel problem is successfully formulated by the DG method in both space and angle [**?**], [**?**]. We present a formulation wherein both space and angle directions are discretized by a DG formulation of a 2D cylindrical radiative transfer problem with axisymmetry.

II. FORMULATION

The general equation of radiative transfer for an emitting, absorbing, and anisotropically scattering medium can be written as,

$$\hat{\boldsymbol{s}} \cdot \nabla I + \beta I = \frac{\sigma_s}{4\pi} \oint_{4\pi} I\left(\boldsymbol{r}, \hat{\boldsymbol{s}}'\right) \Phi\left(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'\right) d\Omega' + q\left(\boldsymbol{r}, \hat{\boldsymbol{s}}\right) \qquad (1)$$

where I is the spectral radiative intensity in the direction $\hat{\boldsymbol{s}}$ and the spatial position $\boldsymbol{r}$, β is the extinction coefficient, σ_s is the scattering coefficient, $\Phi\left(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'\right)$ is known as the phase function, and q is the source term.

The RTE is enforced on the computational domain, $D \times S^2$, where D is the spatial domain and S^2 is the direction, known as the angular domain. To fully discretize the computational domain, each element consists of the spatial mesh and the extrusions of the spatial mesh considered as the angular sub-domain. An illustration of the one angular direction, μ, extrusion from 2D spatial meshes is shown in Fig. 1(a).

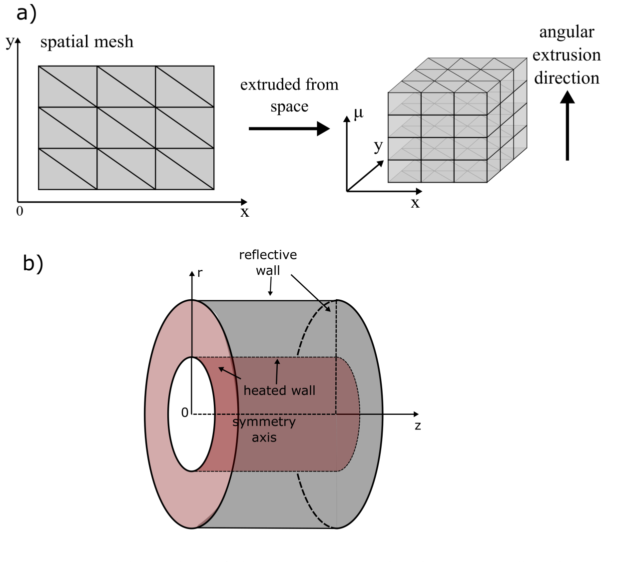

Fig. 1. (a) An illustration of the element extrusion in angular direction. The left plot is 2-D triangular meshes; the right plot is 2-D triangular meshes in space and angular direction meshes. (b) Physical geometry of the cylinder.

In a DG formulation, residuals (errors) must be specified both in the interior and on the boundary of elements. The weighted residual (WR) of the finite element formulations formed by multiplying the RTE (Eqn. 1) by the weight function $\hat{H}$

$$\begin{aligned}&\int_{\mathcal{Q}} \hat{H}\left[\hat{\boldsymbol{s}}\cdot\nabla I+\beta I-q\right]\mathrm{d}V\\ -&\int_{\mathcal{Q}} \hat{H}\left[\frac{\sigma_s}{4\pi}\oint_{4\pi} I\left(\boldsymbol{r},\hat{\boldsymbol{s}}'\right)\Phi\left(\hat{\boldsymbol{s}},\hat{\boldsymbol{s}}'\right)d\Omega'\right]\mathrm{d}V,\\ +&\int_{\partial\mathcal{Q}}\hat{H}\left(I^*-I\right)\hat{\boldsymbol{s}}\cdot n\mathrm{d}A=0\end{aligned} \tag{2}$$

where I^* is the target value in the DG formulation, $\mathcal{Q}$ is the element, $\partial\mathcal{Q}$ is the element boundary, n is the normal vector of the element facet. The weight function $\hat{H}$ and trial solution I are polynomials of order p in both space and angle, interpolated with respect to a local coordinate system. The target value I^* corresponds to the upstream value along the direction of wave propagation where $\hat{\boldsymbol{s}}\cdot n<0$ is the inflow direction and $\hat{\boldsymbol{s}}\cdot n>0$ is the outflow direction.

According to Eqn. 2, the WR of 2D axisymmetric cylindrical RTE, depending on r and z in space and the cosine of the polar angle, $\mu=\cos\theta_s$, in angle, can be derived easily by providing the formulation of $\hat{\boldsymbol{s}}\cdot\nabla I=\sqrt{1-\mu^2}\frac{\partial I}{\partial r}+\mu\frac{\partial I}{\partial z}$ and $\hat{\boldsymbol{s}}=(\sqrt{1-\mu^2},\mu)$.

III. NUMERICAL EXAMPLES

To validate the DG RTE code, The Method of Manufactured Solution (MMS) is used. In the MMS, an exact solution is given as the source term in the DG formulation. If the MS belongs to the space of finite element solution, *e.g.,* when it is a polynomial of order equal or less than that used to interpolate I, the exact solution is recovered. Due to limited space, we will not show the results of MMS here.

Consider two-dimensional steady-state radiation in the absorbing, emitting and anisotropically scattering medium confined between two cylinders with radii, $r_1=1$, $r_2=2$, and height, $h=1$, which is axisymmetric about the z-axis. with the space-dependent scattering coefficient, $\sigma_s(z)=z/h$, and a unit extinction coefficient, $\beta=1$ (in Fig. 1(b)). The phase function for Rayleigh scattering, viz.,$\Phi\left(\mu,\mu'\right)=(3/8)(3-\mu^2-\mu'^2+3\mu^2\mu'^2)$ is used in our example. The constant boundary condition of $\bar{I}=1$ is enforced on the bottom, $z=0$, and the inner wall, $r_1=1$. Reflective boundary conditions are applied on the top, $z=h$, and the outer wall, $r_2=2$.

The basis order in both space and angle is $p=2$. The domain is discretized by 128 triangular elements in space and extruded 20 layers in μ direction. The distribution of radiation intensity in spatial and angular domain is shown in Fig. 2. The wave propagation along z-direction is from the bottom to top when $\mu>0$, and from the top to the bottom when $\mu<0$. Though $z=2$ is set to be reflective wall, The radiation intensity along r-direction seems always to propagate from the left to the right. This is due to $\hat{\boldsymbol{s}}\cdot n$ on the right side of an element is always non-negative. Some parts that are not covered by the wave propagation and reflective wall are caused by Rayleigh scattering, as seen when $\mu=-0.8$.

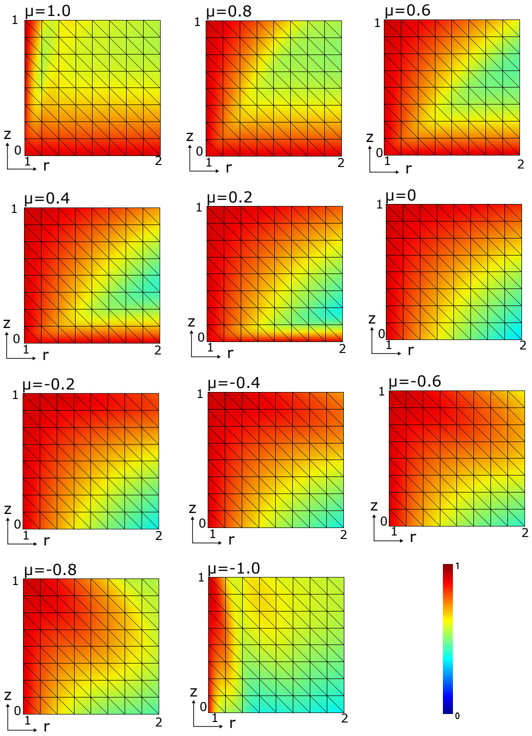

Fig. 2. Contour plot of radiative intensity, I, in $r-z$ space domain with different angle direction, μ.

IV. CONCLUSIONS

This paper has presented a DG method for the numerical solution of 2D cylindrical radiative transfer problems in participating media with axisymmetry. The DG formulation is derived from the general steady-state RTE in both space and angle that can be easily reduced to 2D axisymmetric cylindrical cases. Example cases involving absorbing, emitting, and scattering media were studied using the derived DG formulation. We plan to employ the DG formulation to higher-dimensional RTE in upcoming works.

ACKNOWLEDGEMENT

This material is based upon work supported by the US Air Force Research Laboratory award number FA8650-14-D-1725-0007.

REFERENCES

[1] X. Cui and B. Q. Li, "Discontinuous finite element solution of 2-D radiative transfer with and without axisymmetry," *Journal of Quantitative Spectroscopy and Radiative Transfer*, vol. 96, no. 3-4, pp. 383–407, 2005. [Online]. Available: http://dx.doi.org/10.1016/j.jqsrt.2004.11.007

[2] S. Mudaliar, P. Clarke, and R. Abedi, "Radiative transfer in turbulent flow using spacetime discontinuous Galerkin finite element method," in *Proceedings of 32nd International Union of Radio Science General Assembly & Scientific Symposium (URSI GASS)*, Palais des congres, Montreal, Canada, August 19-26, 2017.

[3] P. L. Clarke, H. Wang, J. M. Garrard, R. Abedi, and S. Mudaliar, "A discontinuous Galerkin method for the solution of one dimensional radiative transfer equation," in *Proceedings of IEEE International Symposium on Antennas and Propagation/USNC-URSI National Radio Science meeting (AP-S/URSI)*, Boston, Massachusetts, USA, July 8-13, 2018.