Analytical Determinant of the Noise Parameter Extraction Matrix and Its Applications

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***Abstract*—We derive an analytical expression of the determinant of the 4×4 noise parameter extraction matrix based on a slight modification of the matrix in [1]. We specialize the derivation to a very useful and practicable case in which one of the impedance points is the center of the Smith chart. The resulting expression is simple and provides very useful guidance for impedance selection. We illustrate this with an example of a wideband noise parameter extraction in the low-frequency radio astronomy band of 50 to 350 MHz.**

I. Introduction

The noise properties of a two-port device such as a low-noise amplifier (LNA) is described by four noise parameters [2]. The measurement of those parameters may be accomplished with four judiciously selected impedance points forming the 4×4 noise parameter extraction matrix [3], [4]. An alternative approach is to use many impedance points and then solving the least-squares problem [1], [2], [5].

Noise parameter extraction with the four-point method is fast and it has been shown that the resulting uncertainties are comparable to the least-squares method [3], [4]. The key here is to avoid matrix singularity and to maximize the linear independence of the rows. To avoid singularity, there are a few obvious cases to avoid [1], [5]. However, how to maximize linear independence is less clear. One example is to numerically search for high matrix determinants [3]; another is to select the points such as the matrix is diagonally dominant [4]. Still if it exists, it is desirable to find a simple analytical expression that closely predicts linear independence. This will inform the trade-off between measurement speed and accuracy, especially in a wideband measurement with a lot of frequency points.

II. Analytical Determinant

We start with the function relating the impedance points with the four noise parameters (a, b, c, d) [1]

$$T_i = a + \frac{b}{1-\gamma_i^2} + c\frac{\gamma_i \cos\theta_i}{1-\gamma_i^2} + d\frac{\gamma_i \sin\theta_i}{1-\gamma_i^2} \tag{1}$$

where γ_i and θ_i are the magnitude and phase, respectively, of the reflection coefficient for measurement $1 \leq i \leq 4$; T_i is the measured noise temperature. We multiply each measurement with $1-\gamma_i^2$

$$T_i' = a(1-\gamma_i^2) + b + c\gamma_i \cos\theta_i + d\gamma_i \sin\theta_i \tag{2}$$

where $T_i' = T_i(1-\gamma_i^2)$.The resulting matrix equation is $\mathbf{t}' = \mathbf{X}'\mathbf{a}$ where $\mathbf{a} = [a, b, c, d]^T$, $\mathbf{t}' = [T_1', \cdots, T_4']^T$ and

$$\mathbf{X}' = \begin{bmatrix} (1-\gamma_1^2) & 1 & \gamma_1\cos\theta_1 & \gamma_1\sin\theta_1 \\ \vdots & \vdots & \vdots & \vdots \\ (1-\gamma_4^2) & 1 & \gamma_4\cos\theta_4 & \gamma_4\sin\theta_4 \end{bmatrix} \tag{3}$$

Note that every entry in $\mathbf{X}'$ is *bounded* by ± 1. This is in contrast to forming the corresponding $\mathbf{X}$ matrix *without* multiplying each row with $1-\gamma_i^2$ where entries in the second to the fourth columns could be *unbounded*. We therefore expect that the maximum $|\det(\mathbf{X}')|$ will be a reasonably small value.

Previous studies have suggested making one measurement with $\gamma = 0$ [3], [4]. Let γ_1=0 such that the first row of $\mathbf{X}'|_{\gamma_1=0}$ is $[1, 1, 0, 0]$. We obtain the determinant $\det(\mathbf{X}'|_{\gamma_1=0})$ by Gaussian elimination and multiplication of the pivots [6]. It can be shown that

$$\begin{aligned} &|\det(\mathbf{X}'|_{\gamma_1=0})| = \\ &\left|\gamma_2\gamma_3\gamma_4\right|\left|\gamma_3\sin\theta_{42} - \gamma_4\sin\theta_{32} - \gamma_2\sin\theta_{43}\right| \end{aligned} \tag{4}$$

where $\theta_{43} = \theta_4 - \theta_3$ and similarly with θ_{42} and θ_{32}. Equation (4) suggests:

1) Regarding the phase, only the *relative* phases matter. The $|\det|$ is invariant with respect to rotation about the center of the Smith Chart.
2) The maximum $|\det|$ of 2.598 occurs for $\gamma_2 = \gamma_3 = \gamma_4 = 1$ and $\theta_{32} = \pm 120°$ and $\theta_{42} = \pm 240°$. The phase difference between points of 120° is in fact the conclusion of the numerical study in [3]. Equation (4) provides the theoretical basis for this as well as for the conjecture that the points must be "well-spread" over the Smith chart [7].

III. Correlation Between the Determinant and Condition Number

We now examine the correlation between $|\det(\mathbf{X}'|_{\gamma_1=0})|$ and the condition number, c, of same matrix. The condition number of a matrix is a measure of the linear independence of the rows and it controls the scaling of uncertainties and errors [6]. For a general square matrix, maximizing the determinant does *not* necessarily maximize linear independence. In fact, we know that $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$ [6]; i.e., maximizing $|\det|$ of a matrix minimizes the $|\det|$ of the inverse. However, due to the multiplication of each row by $1-\gamma_i^2$ and the resulting bounding of the entries of $\mathbf{X}'$ to ± 1, $|\det(\mathbf{X}'|_{\gamma_1=0})|$ is very strongly anti-correlated with its condition number, $c(\mathbf{X}')$: high $|\det|$ predicts low c (good).

Fig. 1 shows the correlation between $|\det(\mathbf{X}'|_{\gamma_1=0})|$ and c using 10^5 random trials with uniform probability over the Smith chart. We see clearly that higher determinants correlate with low condition numbers. If we aim for $|\det|$ of 0.5 or greater, the resulting condition number is between approximately 6 to 12. As a comparison, the condition number of the same matrix with $\gamma_2 = \gamma_3 = \gamma_4 = 0.7$ and phase difference between points of 120° [3] is 6.7. We have also performed similar computations with matrices in [1], [3] in which the entries could be unbounded. They result in significantly more divergence between c and $|\det|$.

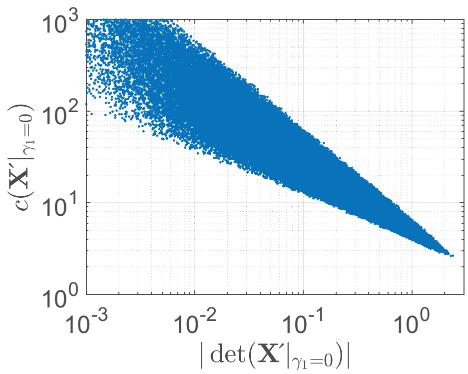

Fig. 1. Correlation of $|\det|$ and the condition number.

IV. Measurement Example

We use the knowledge of (4) to select the fewest probe positions with a Focus Microwave slide screw tuner (rated for 100 MHz to 1 GHz) to cover the low-frequency Square Kilometre Array (SKA-Low) band of 50 to 350 MHz. We find that 6 positions are adequate for this band. Using the impedance loci of the 6 positions, we form two 4×4 matrices to cover the "mid-band" and the "high-band". The resulting determinants are shown in Fig. 2. We see that the (near) singularities in the mid-band at approximately 170 MHz and above 300 MHz are covered by the high-band selection.

We perform noise parameter extraction of the pre-amplifier of a Keysight PXA N9030A at 100 KHz step. The results are reported in Fig. 3 as means and standard deviations computed with 10 MHz window. As expected from the $|\det|$ the (near) singularities of the mid-band are covered by the high-band and vice versa. Also note that away from the singularities, both the means and uncertainties converge well which suggest the measurements are repeatable.

V. Conclusion

We derive the analytical expression for the determinant of a 4×4 noise parameter extraction matrix with one zero reflection coefficient point. The resulting expression is simple and it is a good estimate of the conditioning of the matrix and the resulting measurement uncertainties.

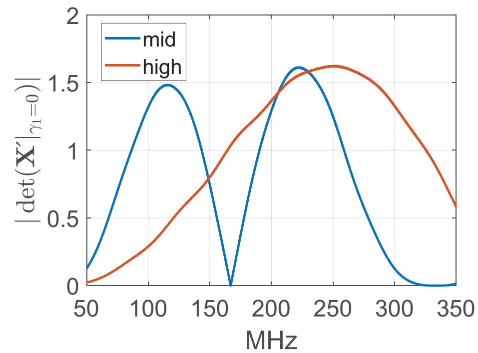

Fig. 2. Determinants of the matrices formed using 6 tuner positions.

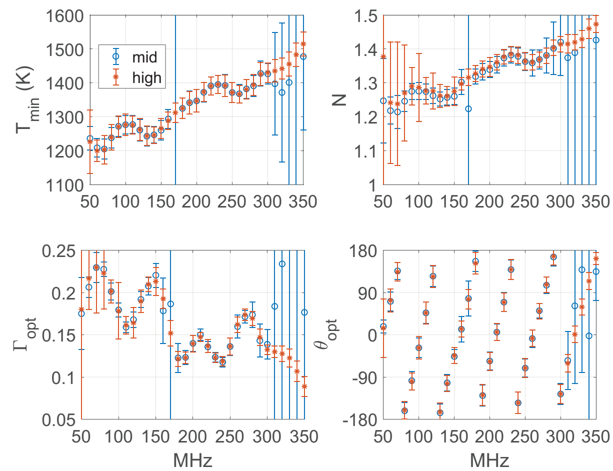

Fig. 3. Noise parameter extraction of the pre-amplifier of a PXA signal analyzer.

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