Singular Integration in BEM by Interpolation: The MFIE Case

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***Abstract*—A fast approach of populating the near (hyper)singular integral interactions in the Magnetic Field Integral Equation (MFIE) Method of Moments (MoM) on flat triangular meshes is proposed. Instead of evaluating each near hypersingular integral between edge-adjacent triangle pairs via specialized integration rules, e.g. singularity subtraction or cancelation, we propose generating a high dimensional *universal* library once-and-for-all, and then using interpolation to evaluate any hypersingular interaction encountered on a MoM matrix. The proposed method is compared to the 1^{st} and 2^{nd} order singularity subtraction method for a couple of MFIE MoM models discretized with Rao-Wilton-Glisson (RWG) basis functions. The proposed method has comparable accuracy to conventional integration approaches but at a ten-fold faster computational cost.**

I. Introduction

Integration between flat patches that share an edge or a vertex in the Boundary Element Method (BEM) solution of the Magnetic Field Integral Equation (MFIE) is a time consuming task that has significant accuracy and reliability implications. Among the most popular methods for evaluating this kind of quadruple (inner/outer) integrals are the singularity subtraction [1] and cancellation [2] approaches, where only the inner integrals are treated using a semi-analytical approach or a singularity reducing change-of-variables, respectively, leaving the outer integrals to numerical quadratures. Recently, the direct evaluation method (DEM) [3] uses ideas from singularity cancellation methods to the full, four dimensional, integral often leading to faster more reliable convergence. Despite those significant advances, reaching a reasonably accuracy (3 to 4 digits) remains significantly slower and less reliable than the respective evaluation of far-field patch interactions.

This paper does not propose another singular integral evaluation method, but an efficient method using either of the aforementioned approaches to populate the near-singular interactions of the BEM impedance matrix. The work will focus on edge-adjacent near singular interactions as they are encountered most often and appear to have more prominent effects on accuracy. For those type of interactions we propose generating a universal library, that is constructed *once-and-for-all* for RWG MFIE BEMs and then use interpolation to expediently recover on-demand any possible edge-adjacent near singular integral interaction on the mesh. Because of the high dimensional nature of integral parametrization, special sampling and interpolation approaches are used to reduce the computational burden. In this work a sparse-grid approach is used for simplicity.

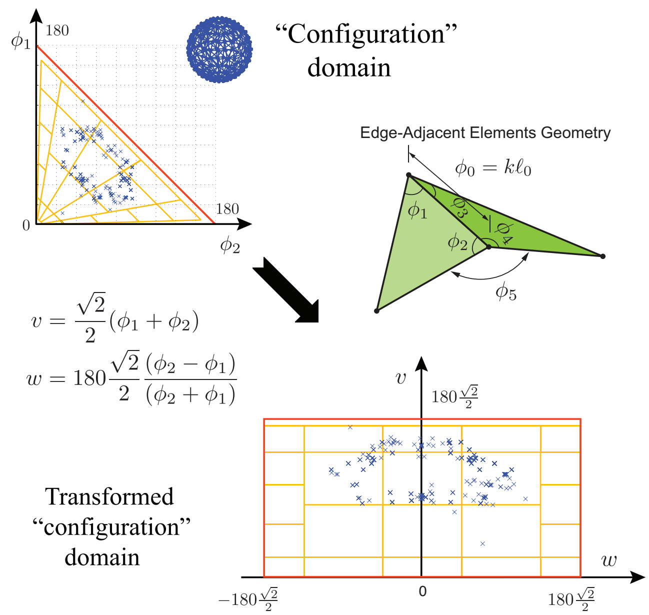

Fig. 1: Overview of the proposed approach. A mesh (shown on top left insert) is considered as a collection of parametric edge-adjacent triangles that are represented as points in a high dimensional parametric space (only 2D is shown for clarity). This configuration space is mapped into a hypercube (parallelogram in the 2D case shown here) where each interaction can be interpolated from a high-dimensional set of pre-sampled points.

II. Approach

The singular entries of the RWG-MFIE BEM matrix are given by:

$$\mathbf{Z}_{mn}(k,T,T') = \frac{1}{2}\int_T \mathbf{f}_m(\mathbf{r})\cdot\mathbf{f}_n(\mathbf{r})\,d\mathbf{r}^2 - \int_T \mathbf{f}_m(\mathbf{r})\cdot\hat{\mathbf{n}}\times\int_{T'}\nabla g_k(\mathbf{r},\mathbf{r}')\times\mathbf{f}_n(\mathbf{r}')d\mathbf{r}'^2 d\mathbf{r}^2, \quad (1)$$

where T and T' are a pair of edge-adjacent observation and source triangles, $\mathbf{r}$ and $\mathbf{r}'$ are the observation and source coordinates, $\mathbf{f}_m$ and $\mathbf{f}_n$ are the the RWG test and trial basis functions respectively, and g_k the free space Helmholtz equation Green's function for wavenumber k. This work will deal with the more troublesome second term in (1). The key in our approach is to recognize that integral $\mathbf{Z}_{mn}(k,T,T')$ in (1) can be parametrized, and written as a function of frequency, the subtended angle ϕ_5 between the edge-adjacent triangle pair, the length of the common edge ℓ_0 and the four angles formed by the common edge and the four other triangle edges denoted as ϕ_1, ϕ_2, ϕ_3, and ϕ_4, as shown in Fig. 1. If the parametrization of ℓ_0 is with respect to its electrical length ϕ_0, it can be shown that the integral is independent of frequency, and thus this parametrization is universal.

The task thus becomes tabulating this universal representation of the edge-adjacent integrals once-and-for-all in a high-dimensional "configuration" domain (hyper-prism). A $2-D$ view of this configuration space is shown in the top left part of Fig. 1, where each point represents an edge-adjacent integral interaction. This configuration domain is discretized with hp interpolation elements that could be non-conforming. To facilitate this process a Duffy transformation, [4]. is used first to convert the configuration domain into a hyper-cube as shown at the bottom left side of Fig. 1. In this domain, edge-adjacent integrals are evaluated at appropriate locations using a very high order singularity subtraction method of 166 Gaussian quadrature points for all numerical integrals. This of course, is performed off-line once-and-for-all generating a library. During runtime this library is accessed and every possible edge-adjacent interaction in a given mesh can be recovered by performing a high-dimensional, yet local interpolation. As will be shown in the results section, this trades some limited memory for a significant run time improvement.

III. Results and Discussion

The proposed approach is compared to the first order singularity subtraction technique, [1] in terms of accuracy and computational speed. Figs 2 and 3 show the relative error histograms corresponding to all edge-adjacent near singular BEM $\mathbf{Z}$ matrix entries for a drone aircraft mesh of 53,004 triangles and a fighter jet intake cavity mesh of 26,563 triangles. In both plots blue bars correspond to the proposed method, while the red correspond to a reasonable $(7,7,73)$ singularity subtraction rules, and green correspond to a very high-order one $(73,73,73)$. Those integration triplets denote the number of Gaussian quadrature points in the inner and outer rules of the subtracted term, followed by the outer quadrature rule of the analytical term. Note that in these plots the reference results have been obtained by a singularity subtraction rule of $(166,166,166)$. In both cases all methods show average error of about $2-3$ decimal digits, while the maximum error, corresponding to triangles with very poor quality, can be rather large for all three methods. In the case of the intake cavity problem the proposed approach has more interactions with good errors (below 3 decimal digits), while it also has less interaction with bad error (above 2 decimal digits). Computational statistics for both problems are summarized in Table I, where a comparison of L_2 and L_∞ errors and computation time is given. In this table boldface numbers indicated best performance. This table shows that all methods are close in accuracy, while the proposed method is about 10 times faster.

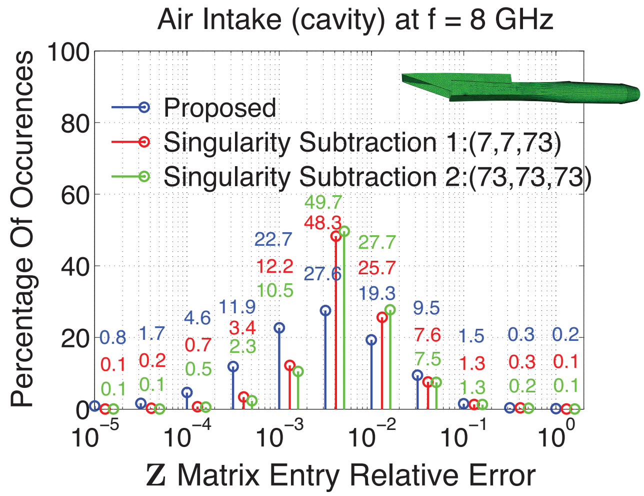

Fig. 2: Relative error histogram for a fighter jet intake cavity. Comparison of proposed method with two, 1^{st} order singularity subtraction rules.

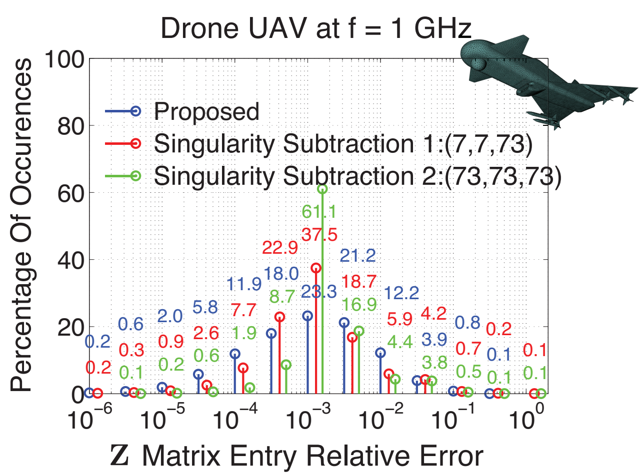

Fig. 3: Relative error histogram for a drone aircraft.

TABLE I: Computational Statistics Summary

mesh		Proposed	SS 1	SS 2
intake	time	**56** [s]	551 [s]	1,835 [s]
$N: 26563$	L_2	$8.5 \cdot 10^{-3}$	$\mathbf{4.8 \cdot 10^{-3}}$	$5.0 \cdot 10^{-3}$
$f = 8$GHz	L_∞	$9.1 \cdot 10^{0}$	$5.3 \cdot 10^{-0}$	$\mathbf{5.1 \cdot 10^{-0}}$
drone	time	**123** [s]	1,173 [s]	3,812 [s]
$N: 53004$	L_2	$8.5 \cdot 10^{-3}$	$5.4 \cdot 10^{-3}$	$\mathbf{5.2 \cdot 10^{-3}}$
$f = 1$GHz	L_∞	$\mathbf{1.5 \cdot 10^{0}}$	$2.4 \cdot 10^{0}$	$2.4 \cdot 10^{0}$

N: Number of interactions, SS 1,2: singularity subtraction rule 1, 2, L_2 : average error, L_∞ : maximum error.

References

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