Derivation of the Fundamental Solution of Dirac Equation Using Green Functions

Morteza Shahpari and Andrew Seagar
School of Engineering, Griffith University

There are variety of different methods to solve Maxwell's equations for a given geometry. One of the conventional methods is to use the vector algebra, and Green theorem that are properly developed for a given geometry. An alternative method is to use Clifford algebra and the Cauchy theorem. Under the framework of the Clifford algebra, one casts Maxwell's four equations into a single first order differential equation which is referred to as the Dirac equation $\mathcal{D}\mathcal{F} = \mathcal{S}$. This method is called Clifford-Cauchy-Dirac (CCD). A solution of the Dirac equation with an infinitesimal source is often referred to as the fundamental solution F_k which are used to find the Cauchy kernels in the CCD method. So far, only the fundamental solution for free space medium is given in the literature. Here, we propose a simple method to get the fundamental solutions $\mathcal{F}$ from the equivalent Green's functions.

Assuming $\boldsymbol{G}_e$ and $\boldsymbol{G}_m$ represent the appropriate Green's functions for electric and magnetic fields respectively while they satisfy boundary conditions of the problem and $\boldsymbol{J} = \delta(\boldsymbol{R} - \boldsymbol{R'})$, we have the following Helmholtz equations:

$$\nabla \times \nabla \times \boldsymbol{G}_e - k^2 \boldsymbol{G}_e = \boldsymbol{J}\delta(R - R'), \qquad \nabla \times \nabla \times \boldsymbol{G}_m - k^2 \boldsymbol{G}_m = \nabla \times \boldsymbol{J}\delta(R - R') \tag{1}$$

Depending on the geometry and boundary conditions of a specific problem, one might choose to solve either equations of (1) and then use $\nabla \times \boldsymbol{G}_e = \boldsymbol{G}_m$ or $\nabla \times \boldsymbol{G}_m = \boldsymbol{I}\delta(R - R') + \boldsymbol{G}_e$ to find the other. For example, solving for rectangular waveguide is often done by first solving (1) for the magnetic dyadic Green's function $\boldsymbol{G}_m$.

In the CCD framework, Maxwell equations are casted into a simple first order differential equation:

$$\mathcal{D}\mathcal{F} = \mathcal{S} \tag{2}$$

where $\mathcal{D} = \frac{\partial}{\partial x}e_1 + \frac{\partial}{\partial y}e_2 + \frac{\partial}{\partial z}e_3 + ke_0$, $\mathcal{F} = \sqrt{\mu}\boldsymbol{H}\sigma - j\sqrt{\epsilon}\boldsymbol{E}e_0$ and $\mathcal{S} = \sqrt{\mu}\boldsymbol{J} + \frac{j}{\sqrt{\epsilon}}\rho e_0$. Here, we use the Clifford rule as $e_i e_j + e_j e_i = -2\delta_{ij}$ and σ is defined as $-e_1 e_2 e_3$.

Since the electric and magnetic Green's functions $\boldsymbol{G}_e$ and $\boldsymbol{G}_m$ can be substituted in to Maxwell's equations and satisfy the same sort of boundary conditions, they also satisfy (2) which is another form of Maxwell's equations.

$$\mathcal{F} = \sqrt{\mu}\boldsymbol{G}_m\sigma - j\sqrt{\epsilon}\boldsymbol{G}_e e_0, \tag{3}$$

$$F_k = \mathcal{F}\boldsymbol{J}^{-1} \tag{4}$$

where the vector inversion is simply done by $\boldsymbol{J^{-1}} = -\frac{\boldsymbol{J}}{|\boldsymbol{J}|^2}$. Equation (4) is a significant result which empowers one to easily find fundamental solution of any set of boundary conditions as long as the Green's functions of the problem are already known. In the presentation, illustrative examples of fundamental solution of rectangular and circular waveguides will be provided.