Calderon's preconditioning for the EFIE without the barycentric elements

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***Abstract*—A new preconditioning for the electric field integral equation (EFIE) based on the Calderon formulae is discussed. Although well-conditioned EFIEs, which do not suffer from the dense discretisation breakdown, have been widely studied, many formulations proposed in previous researches require the implementation of the Buffa-Christiansen (BC) basis functions or the loop-star decomposition. In this paper, we propose an implementation of the Calderon preconditioning, which is discretised with the Rao-Wilton-Glisson (RWG) basis functions and the piecewise constant functions. We also show that the Calderon preconditioning implemented in this way also solves the so-called "low-frequency breakdown".**

I. Introduction

The EFIE [1], which is one of formulations of integral equations for solving electromagnetic scattering problems, is known to have several problems from the numerical point of view. One of the problems, called "dense discretisation breakdown", causes slow convergence when the EFIE is solved with an iterative linear solver such as the generalised minimal residual method (GMRES). Many formulations of well-conditioned EFIEs, which do not suffer from the dense discretisation breakdown, have been widely studied [2], [3], [4], [5]. These formulations, however, require the implementation of the BC basis functions, which are defined on the barycentric elements having six times more triangular elements than an original mesh used for discretising the EFIE, or the implementation of the loop-star decomposition.

Another numerical problem on the EFIE is called "low-frequency breakdown" [6]. This breakdown causes bad accuracy when kh is too small, where k is the wave number and h is typical size of the mesh. It is widely known that simple application of the Calderon preconditioning to the EFIE cannot solve the low-frequency breakdown. Hence many solution methods for the low-frequency breakdown and combination of these methods with the well-conditioned EFIE stated above have been studied.

In this paper, we propose a new preconditioning method for the EFIE which is based on the Calderon formulae but does NOT require the use of the barycentric elements and the loop-star decomposition. We show that the preconditioned electric field integral operator (EFIO) with the Calderon formulae is represented by four integral operators, which consists of parts of the EFIO and the magnetic field integral operator (MFIO) and can be discretised with the RWG basis functions and the piecewise constant function. We also note that the well-conditioned EFIE implemented in this way also solves the low-frequency breakdown.

II. Formulation

We consider a domain Ω occupied by a PEC with smooth boundary Γ. This PEC is illuminated by an incident wave $(\boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{H}^{\mathrm{inc}})$. In order to solve this problem we consider the EFIE as follows:

$$\mathrm{i}\omega\mu Q\boldsymbol{j} = \boldsymbol{E}^{\mathrm{inc}} \times \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{j}$ is the unknown electric current on Γ, Q is the EFIO:

$$Q\boldsymbol{j} := \boldsymbol{n} \times \int_\Gamma \left\{ G(\boldsymbol{x}-\boldsymbol{y}) + \frac{1}{k^2}\nabla\nabla G(\boldsymbol{x}-\boldsymbol{y}) \right\} \boldsymbol{j}(\boldsymbol{y}) \mathrm{d}S_y,$$

$k = \omega\sqrt{\varepsilon\mu}$, ε, μ are the permittivity and permeability and G is the Green function of the Helmholtz equation:

$$G(\boldsymbol{x}-\boldsymbol{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

The integral operator Q satisfies the following Calderon's formulae:

$$k^2Q^2 = \frac{\mathcal{I}}{4} + P^2, \tag{2}$$

where

$$P\boldsymbol{m} := \boldsymbol{n} \times \int_\Gamma \nabla G(\boldsymbol{x}-\boldsymbol{y}) \times \boldsymbol{m} \mathrm{d}S_y.$$

Calderon's formulae in (2) implies that the integral operator k^2Q^2 is "well-conditioned" since P^2 is compact. Hence now, we multiply the both hand sides of the integral equation in (1) by $-\mathrm{i}\omega\varepsilon Q$ and try to discretise k^2Q^2.

III. Discretisation of the square of Q

We first decompose Q as

$$Q = S + \frac{1}{k^2}N$$

where

$$S = \boldsymbol{n} \times \int_\Gamma G\boldsymbol{j}\mathrm{d}S_y, \quad N = \boldsymbol{n} \times \int_\Gamma \nabla\nabla G\boldsymbol{j}\mathrm{d}S_y$$

Then,

$$k^2Q^2 = k^2(S + \frac{1}{k^2}N)Q = k^2SQ + NS$$

since $N^2 = 0$. We discretise the two operators k^2SQ and NS in different ways. First, the operator k^2SQ can be discretised with only the RWG basis functions as follows:

$$k^2T^{-1}S\widetilde{T}^{-1}Q$$

where

$$(S)_{ij} := \langle \boldsymbol{t}_i, S(\boldsymbol{n} \times \boldsymbol{t}_j)\rangle_{L^2(\Gamma)}, \quad (Q)_{ij} := \langle \boldsymbol{n} \times \boldsymbol{t}_i, Q\boldsymbol{t}_j\rangle_{L^2(\Gamma)},$$
$$(T)_{ij} := \langle \boldsymbol{t}_i, \boldsymbol{t}_j\rangle_{L^2(\Gamma)} = \langle \boldsymbol{n} \times \boldsymbol{t}_i, \boldsymbol{n} \times \boldsymbol{t}_j\rangle_{L^2(\Gamma)} =: (\widetilde{T})_{ij},$$

$\boldsymbol{t}_i$ is the RWG basis function of ith element and

$$\langle u, v\rangle_{L^2(\Gamma)} = \int_\Gamma \bar{u} \cdot v \mathrm{d}S$$

The second operator NS cannot be discretised in a similar way of the first operator k^2SQ since N is a hyper singular operator. Thus we move one derivative in N to S by taking integration by parts:

$$\begin{aligned} NS\boldsymbol{j} &= \boldsymbol{n} \times \int_\Gamma \nabla\nabla G \left\{ \boldsymbol{n} \times \int_\Gamma G\boldsymbol{j}\mathrm{d}S \right\} \mathrm{d}S \\ &= \boldsymbol{n} \times \int_\Gamma \nabla G \left\{ \nabla \cdot \boldsymbol{n} \times \int_\Gamma G\boldsymbol{j}\mathrm{d}S \right\} \mathrm{d}S \\ &= \boldsymbol{n} \times \int_\Gamma \nabla G \left\{ -\boldsymbol{n} \cdot \int_\Gamma \nabla G \times \boldsymbol{j}\mathrm{d}S \right\} \mathrm{d}S \\ &= -DM^* \end{aligned}$$

where

$$Du := \boldsymbol{n} \times \int_\Gamma \nabla^x G(\boldsymbol{x} - \boldsymbol{y}) u \mathrm{d}S_y$$
$$M^*\boldsymbol{j} := \boldsymbol{n} \cdot \int_\Gamma \nabla^x G(\boldsymbol{x} - \boldsymbol{y}) \times \boldsymbol{j}\mathrm{d}S_y$$

Note that M^* maps a vector-valued function to a scalar function while D maps a scalar function to a vector-valued function. The operator $-DM^*$ can be discretised as follows:

$$T^{-1}DT_c^{-1}M^*$$

where

$$(D)_{ij} = \langle \boldsymbol{t}_i, Dq_j\rangle_{L^2(\Gamma)}, \quad (M^*)_{ij} = \langle q_i, M^*\boldsymbol{t}_j\rangle_{L^2(\Gamma)},$$
$$(T_c)_{ij} = \langle q_i, q_j\rangle_{L^2(\Gamma)},$$

and q_i is the piecewise constant function defined on ith element.

IV. Numerical example

We consider a spherical PEC with the radius 0.25 illuminated by a plane wave. We compare the EFIE preconditioned by the proposed preconditioning method in the previous section, the standard Calderon preconditioning, which utilises the RWG and BC basis functions, and no preconditioning. The linear equation is solved by the GMRES with the error tolerance 10^{-5}. Note that, in this example, all matrices are directly calculated without any fast methods in order to simplify the implementation. Fig.1 shows the iteration number of the GMRES for the three kinds of preconditioning. The iteration number of the proposed method is the least

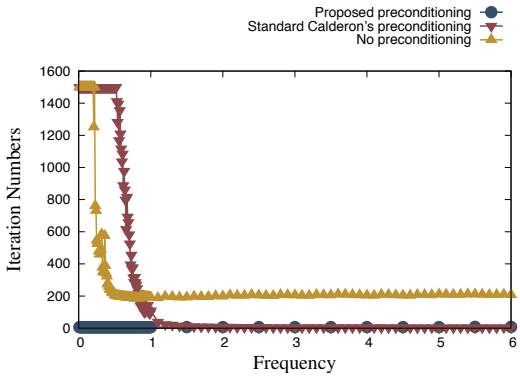

Fig. 1. The iteration number of the GMRES.

among the compared three preconditionings for all frequencies tested in this example. Also, we can see that the proposed method converges fast even in small frequencies. This is because $N^2 = 0$ is calculated analytically in the proposed method while, in the standard Calderon preconditioning, N^2 is estimated numerically and it diverges when the frequency is small.

V. Conclusions

A Calderon preconditioning for the EFIE without the barycentric elements was proposed in this paper. By a simple numerical example, we have verified that the proposed preconditioning method can efficiently decrease the iteration number without suffering from the low-frequency breakdown. Our future plan is to implement a fast method such as the fast multipole method (FMM) and test the proposed method in more large problems.

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