A Linear-Complexity Randomized Butterfly Scheme for Direct Integral Equation Solvers

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Butterfly schemes, also known as multilevel matrix decomposition algorithms, constitute efficient schemes for compressing matrices associated with highly oscillatory operators frequently arising in integral equation (IE) solvers, spherical harmonic transforms, Fourier integrals, data science applications, etc. We recently developed butterfly-based direct IE solvers that rapidly construct butterfly-compressed forward and inverse IE operators for analyzing scattering from both 2D and 3D, perfectly electrically conducting and homogenous dielectric objects involving millions of unknowns (Liu et. al., *IEEE Antenn. Wireless Propag. Lett.*, 2017; Guo et. al., *IEEE Trans. Antennas Propag.*, 2017). These solvers leverage both *iterative* and *non-iterative* randomized schemes to construct butterfly-compressed approximations of matrices formed by adding and multiplying already butterfly-compressed matrices. However, both randomized schemes have their limitations. The iterative scheme only exhibits rapid convergence rates for butterflies with few levels (Guo et. al., in *URSI Digest*, 2014); the non-iterative scheme, though capable of reconstructing arbitrary-level butterflies, requires more CPU operations (Liu et. al., in *URSI Digest*, 2016). As a result, these direct solvers exhibit $O(N\log^2 N)$ memory and $O(N^{1.5}\log N)$ CPU complexities with N representing the number of unknowns. To further reduce the CPU complexity of butterfly-based direct solvers, faster randomized butterfly schemes are called for.

In this work, we propose a new non-iterative randomized butterfly scheme for butterfly-based direct solvers provided that target blocks can be rapidly applied to arbitrary vectors. The proposed scheme relies on an additional rank property of the butterfly substructures and utilizes hierarchical structured random vectors to reconstruct blocks in the desired factorization. The proposed scheme represents a significant improvement over its predecessors. (i) Unlike the previous iterative scheme, the proposed scheme allows for the successful reconstruction of arbitrary-level butterfly-compressed blocks with overwhelmingly high probability. (ii) Unlike the previous non-iterative scheme that requires $O(n^{0.5})$ random vectors, the proposed scheme only requires $O(\log n)$ random vectors where n represents the block dimensions. The memory and computational costs of reconstructing a butterfly-compressed block are theoretically proven and numerically validated to be $O(n\log n)$ and $O(n\log^2 n)$, respectively. The abovementioned butterfly-based direct solvers, when enhanced by this optimal-complexity randomized butterfly scheme, can achieve almost linear CPU complexity. The computational efficiency of the butterfly-based direct solver leveraging the new randomized scheme is demonstrated through its application to scattering problems involving several million unknowns.