Numerical Investigation of the Influence of Stokes Lines on Creeping Wave Propagation

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***Abstract*—Creeping wave propagation is explored within the framework of the well-known high-frequency asymptotic solution to the canonical problem of a line source radiating in the presence of a perfectly conducting circular cylinder of electrical radius ka. Employing the Watson transform to the appropriate eigenfunction expansion solution, the creeping wave propagation constant ν, for a given ka, are the zeros of $\mathrm{H}_\nu^{(2)}(ka) = 0$ (or $\mathrm{H}_\nu^{(2)'}(ka) = 0$), for electric (or magnetic) line sources, respectively. For electrically large cylinders, $|ka| \gg 1$ and $|\nu| \sim ka$, it is customary to replace the Hankel functions by their Airy representations. The (composite) asymptotic form of the Airy function contains dominant and subdominant terms which switch roles across the Stokes lines. In this work, a new algorithm is proposed to calculate the creeping wave propagation constant based on uniform asymptotic expansion of the Airy function across the Stokes lines.**

I. Introduction

The high-frequency electromagnetic field in the geometrical shadow region of a canonical convex surface can be expressed via the exact eigenfunction solutions [1], [2], having practical applications to conformal antennas [3]. It is well known that for the simplest canonical geometry of a circular cylinder, computations utilizing the exact eigenfunction solution becomes exceedingly challenging for electrically large cylinders ($ka \to \infty$). In such specific situations, the eigenfunction series is converted to a contour integral [4]–[6], that is evaluated via residues or by other sophisticated asymptotic methods [7], [8]. The subject of this investigation is to study the numerical evaluation of the shadow region (or *creeping wave*) field obtained by asymptotic methods.

Of the many high-frequency asymptotic methods, the *Uniform Theory of Diffraction*(UTD) [9]– [12] has found many practical applications. The radiation from creeping wave is a dominant contributor to the scattered field in the shadow region [13]. According to the UTD ansatz, the creeping wave travels along the shortest path (geodesic) between two points on a convex surface, undergoing an exponential attenuation in the direction of geodesic path of propagation.

Analysis of creeping wave propagation has been extensively studied earlier by various authors [14]- [17]. Employing the Watson transform [4], [18], and using the Airy approximation to the Hankel function [19], the creeping wave propagation constants are determined. Recent investigations on creeping wave scattering by a PEC circular cylinder excited by a line source [20], [21] utilized a more general form of the Airy function than [19]. These generalized asymptotic forms are obtained from recent techniques [22]– [29] that essentially uniformize the Airy function with large complex argument across the Stokes lines, yielding a composite asymptotic expansion. The main purpose of this investigation is to revisit the numerical calculation of creeping wave propagation constants using the uniform asymptotic form of the Airy function [27, p. 80, Eq. (1.7.3)]. This specific form corrects the asymptotic representation when Stokes lines are crossed, and is a generalization of the one used in [19, Eq.(19)].

The limitation of the approach proposed here is that it strictly applies to perfectly conducting surfaces. A consideration is that although the present formulation is directly relevant for a line source (2-D) problem, one can directly extend the concepts to the 3-D problem [1], [2], [7].

In what follows, section II briefly summarizes the new concept of *hyperasymptotics* from an earlier exposition [30] of the same within the framework of the Airy function of a complex argument. This is followed by description of an algorithm primarily developed from [20], [21] in section III. Numerical results from this algorithm are currently in progress and shall be furnished at the time of presentation.

II. Airy Function and Hyperasymptotics

For large arguments, the Fock approximation [15], to the Hankel function $\mathrm{H}_\nu^{(2)}(\nu z)$ gives its asymptotic representation [22, p. 232–233, Eq. (10.20.6)] that reads,

$$\mathrm{H}_\nu^{(2)}(z\nu) \approx 2e^{+\jmath\pi/3}\mathcal{F}(\zeta,z)\left(\frac{\mathrm{Ai}(\mu)}{\nu^{1/3}}\sum_{k=0}^{+\infty}\frac{\mathcal{A}_k(\zeta)}{\nu^{2k}} + \frac{e^{-2\jmath\pi/3}\mathrm{Ai}'(\mu)}{\nu^{5/3}}\sum_{k=0}^{+\infty}\frac{\mathcal{B}_k(\zeta)}{\nu^{2k}}\right). \quad (1)$$

In (1), $z = \frac{ka}{\nu}$ and the explicit forms for the function $\mathcal{F}(\zeta, z)$, and the coefficients $\mathcal{A}_k(\zeta)$ and $\mathcal{B}_k(\zeta)$, can be found in [22] and are omitted for brevity. The transformation variable ζ is given by

$$\frac{2}{3}\zeta^{\frac{3}{2}} = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2}, \text{ if } 0 < z \le 1,$$
$$\frac{2}{3}(-\zeta)^{\frac{3}{2}} = \sqrt{z^2-1} - \cos^{-1}\left(\frac{1}{z}\right), \text{ if } 1 \le z < \infty. \quad (2)$$

Finally $\mathcal{A}_0(\zeta) = 1$, and the higher order coefficients can be through recursion. (The asymptotic form of $\mathrm{H}_\nu^{(2)'}(z\nu)$ is also available in [22, p. 233].)

Retaining the dominant ($k = 0$) terms only, one can obtain from (1), the relation:

$$\begin{aligned} \frac{\mathrm{Ai}(\mu)}{\mathrm{Ai}'(\mu)} &\approx -e^{-\jmath 2\pi/3}\nu^{-4/3}\mathcal{B}_0(\zeta) \text{ where,} \\ \mu &= e^{-\jmath 2\pi/3}\nu^{2/3}\zeta, \text{ and,} \\ \mathcal{B}_0(\zeta)\Bigg|_{\zeta\to\infty} &\approx -\frac{5}{48\zeta^2}. \end{aligned} \tag{3}$$

Equation (3) is to be solved for determining the zeros of the Hankel function. The Airy function & it's 1st derivatives are analytic functions that admit convergent series expansions [24, p. 91, Eq. (4.23)]. However when its asymptotic expansions are desired, then the algebraic process introduces different multivalued functions that are valid within limited regions of the complex plane. So, one asymptotic expansion cannot be analytically continued to the other and is the cause of the Stokes phenomenon. Across the Stokes lines of the Airy function, which occur at rays emanating at angles $\pm\frac{2\pi}{3}$, there is a severe numerical discontinuity in the two asymptotic expansions. This is corrected through a uniformization and is described in [27]. From [20], the uniform asymptotic expansion of the Airy function can be obtained as:

$$\begin{aligned} \mathrm{Ai}(z) &\approx \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}}[f(z) + \jmath\mathcal{S}(z)f^{-1}(z)], \\ \mathrm{Ai}'(z) &\approx \frac{-1}{2\sqrt{\pi}} z^{+\frac{1}{4}}[f(z) - \jmath\mathcal{S}(z)f^{-1}(z)], \\ f(z) &= \exp(-\frac{2}{3}z^{\frac{3}{2}}), \\ \mathcal{S}(z) &= \frac{1}{2} + \frac{1}{2}\mathtt{erf}[\sigma(z)], \\ \sigma(z) &= \frac{\Im m[\mathcal{F}(z)]}{\sqrt{2\Re e[\mathcal{F}(z)]}}, \\ \mathcal{F}(z) &= -\frac{4}{3}z^{\frac{3}{2}}. \end{aligned} \tag{4}$$

Across the Stokes lines the term $\mathcal{S}(z)$ takes a value of 1. The *subdominant* $f^{-1}(z)$, and *dominant* $f(z)$, terms in (4) switch roles across the Stokes lines. This phenomenon significantly contributes to the location of the complex zeros of the Airy function [23], [28], [29] and hence the solutions to (3) & (2).

III. Proposed Algorithm and Results

Using (4), (3) will be solved numerically for the creeping wave propagation constant ν and the results are anticipated to be furnished at the time of the presentation for various values of ka.

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