A Modified Admissibility Criterion for $\mathcal{H}$-Matrix Based Integral-Equation Solvers

[1]Anton Menshov [2]Vladimir Okhmatovski, and [1,3]Ali E. Yılmaz
[1]Department of Electrical and Computer Engineering, The University of Texas at Austin, Austin, TX, USA
[2]Department of Electrical and Computer Engineering, University of Manitoba, Winnipeg, MB, Canada
[3]Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX, USA
anton.menshov@utexas.edu, vladimir.okhmatovski@umanitoba.ca, ayilmaz@mail.utexas.edu

***Abstract*—An alternative matrix block hierarchy construction process is presented for $\mathcal{H}$-matrix based fast direct solvers. In the proposed method, the traditional distance-based criterion for admitting a given block of the impedance matrix is augmented by also evaluating the admissibility of its children blocks.**

I. Introduction

Fast direct integral-equation methods based on the $\mathcal{H}$-matrix framework [1] are becoming increasingly more popular for electromagnetic simulations; e.g., they have recently been proposed for hydraulic fracture diagnosis [2], network parameter extraction [3], and predictive modeling of electronic packages [4]. Such methods can outperform traditionally used fast iterative methods if the problem of interest requires solutions for multiple excitations and/or the number of iterations needed for convergence of the iterative linear system solver is relatively large [2], [4].

$\mathcal{H}$-matrix approximation of the dense impedance matrix arising from the method of moments discretization of integral equations relies on the matrix block low-rank structure. Indeed, the impedance matrix block corresponding to an interaction between well-separated subdomains of the region of interest is expected to have an effective rank that is significantly smaller than the number of basis/testing functions, i.e., matrix block column and row dimensions. The choice of the block hierarchy greatly affects the performance of the fast method and required memory. Commonly, a simple partitioning and purely geometry-based admissibility criterion are used to determine if basis and testing subdomains are well-separated relative to their sizes [1]–[5].

A better block hierarchy can be potentially achieved by estimating a block's rank in advance; however, the construction process must remain efficient with respect to the other parts of the algorithm, so a fast rank estimation algorithm must be used in this case, e.g., [6]. This article presents a modification of the traditional admissibility criterion that improves the obtained block hierarchy by unifying certain matrix blocks based on their child-parent relations.

II. Presented Work

The construction of the block hierarchy can be described in two steps. First, the solver bisects the geometry hierarchically into disjoint subdomains until the leaf size is reached forming a cluster tree that has at the ℓ th level $\mathcal{O}(2^\ell)$ subdomains. Next, the solver constructs the block hierarchy by recursively checking if an interaction $(\mathcal{I}_i^{(\ell)} \times \mathcal{J}_i^{(\ell)})$ between a cluster of basis functions $\mathcal{J}_j^{(\ell)}$ and cluster of testing functions $\mathcal{I}_i^{(\ell)}$ at the ℓ th level is potentially low-rank, according to a certain admissibility criterion. If the interaction $(\mathcal{I}_i^{(\ell)} \times \mathcal{J}_j^{(\ell)})$ is admissible, the corresponding impedance matrix block is approximated in a low-rank form $\mathbf{Z}_{i,j}^{(\ell)} \approx \mathbf{A}_{i,j}^{(\ell)} \mathbf{B}_{i,j}^{(\ell)H}$, otherwise the four children interactions $(\mathcal{I}_{i,1}^{(\ell+1)} \times \mathcal{J}_{j,1}^{(\ell+1)})$, $(\mathcal{I}_{i,1}^{(\ell+1)} \times \mathcal{J}_{j,2}^{(\ell+1)})$, $(\mathcal{I}_{i,2}^{(\ell+1)} \times \mathcal{J}_{j,1}^{(\ell+1)})$, and $(\mathcal{I}_{i,2}^{(\ell+1)} \times \mathcal{J}_{j,2}^{(\ell+1)})$ are considered. The process repeats until the leaf level is reached.

In [2]–[5], the admissibility of the interactions is determined by a simple distance-based criterion:

$$\begin{gathered} \mathrm{adm}(\mathcal{I}_i^{(\ell)} \times \mathcal{J}_j^{(\ell)}) \;=\; \text{true} \\ \Leftrightarrow \\ \min(\mathrm{diam}(\mathcal{B}_{\mathcal{I}_i}^{(\ell)}), \mathrm{diam}(\mathcal{B}_{\mathcal{J}_j}^{(\ell)})) \le \eta \; \mathrm{dist}(\mathcal{B}_{\mathcal{I}_i}^{(\ell)}, \mathcal{B}_{\mathcal{J}_j}^{(\ell)}),\ \eta > 0 \end{gathered} \tag{1}$$

Here, $\mathrm{diam}(\cdot)$ denotes the diameter of the box bounding the source/observer cluster and $\mathrm{dist}(\cdot,\cdot)$ is the minimum distance between them. Notice that calculation of the minimum distance between the basis/testing function clusters $\mathrm{dist}(\mathcal{I}_i^{(\ell)}, \mathcal{J}_j^{(\ell)})$ rather than their bounding boxes in (1) is non-trivial for general non-convex domains. In (1), η is a tuning parameter that controls the number of blocks at each level ℓ that are approximated using low-rank form.

While the above process rapidly segregates the block hierarchy into admissible and inadmissible blocks, it can fail to identify compressible blocks or result in matrix blocks that are stored inefficiently, e.g., some blocks may have to be partitioned further or unified with their neighbors to achieve optimal performance. To demonstrate this, consider an interaction

$(\mathcal{I}_i^{(\ell)} \times \mathcal{J}_j^{(\ell)})$ that fails the admissibility criterion (1) but is such that all its children $(\mathcal{I}_{i,1}^{(\ell+1)} \times \mathcal{J}_{j,1}^{(\ell+1)})$, $(\mathcal{I}_{i,1}^{(\ell+1)} \times \mathcal{J}_{j,2}^{(\ell+1)})$, $(\mathcal{I}_{i,2}^{(\ell+1)} \times \mathcal{J}_{j,1}^{(\ell+1)})$, and $(\mathcal{I}_{i,2}^{(\ell+1)} \times \mathcal{J}_{j,2}^{(\ell+1)})$ meet the criterion for the same value of η. Next, consider a partitioned matrix of the form

$$\underbrace{\mathbf{Z}_{i,j}^{(\ell)}}_{m\times n} = \begin{bmatrix} \overbrace{\mathbf{Z}_{1,1}^{(\ell+1)}}^{m_1\times n_1} & \overbrace{\mathbf{Z}_{1,2}^{(\ell+1)}}^{m_1\times n_2} \\ \underbrace{\mathbf{Z}_{2,1}^{(\ell+1)}}_{m_2\times n_1} & \underbrace{\mathbf{Z}_{2,2}^{(\ell+1)}}_{m_2\times n_2} \end{bmatrix}, \quad \begin{matrix} m_1+m_2=m, \\ n_1+n_2=n, \end{matrix} \tag{2}$$

where $\mathbf{Z}_{1,1}^{(\ell+1)}$, $\mathbf{Z}_{1,2}^{(\ell+1)}$, $\mathbf{Z}_{2,1}^{(\ell+1)}$, and $\mathbf{Z}_{2,2}^{(\ell+1)}$ are the matrix blocks representing the children's interactions, respectively. It can be shown that the rank of the parent block is bounded by [7]

$$\begin{aligned} \operatorname{rank}(\mathbf{Z}_{i,j}^{(\ell)}) \le\ & \operatorname{rank}(\mathbf{Z}_{1,1}^{(\ell+1)}) + \operatorname{rank}(\mathbf{Z}_{1,2}^{(\ell+1)}) \\ & + \operatorname{rank}(\mathbf{Z}_{2,1}^{(\ell+1)}) + \operatorname{rank}(\mathbf{Z}_{2,2}^{(\ell+1)}) \\ & + \min\left[\operatorname{rank}(\mathbf{Z}_{1,2}^{(\ell+1)}), \operatorname{rank}(\mathbf{Z}_{2,1}^{(\ell+1)})\right] \end{aligned} \tag{3}$$

For simplicity, let $m_{1,2} = n_{1,2} = m/2 = n/2$ and $\operatorname{rank}(\mathbf{Z}_{1,1}^{(\ell+1)}) = \operatorname{rank}(\mathbf{Z}_{1,2}^{(\ell+1)}) = \operatorname{rank}(\mathbf{Z}_{2,1}^{(\ell+1)}) = \operatorname{rank}(\mathbf{Z}_{2,2}^{(\ell+1)}) = k$. Then, the storage of the matrix block representing an interaction $(\mathcal{I}_i^{(\ell)} \times \mathcal{J}_j^{(\ell)})$ in $\mathbf{AB}^H$-form at $(\ell+1)$th level requires $\mathcal{O}(4mk)$ memory, whereas storing the interaction at the level ℓ would at most require $\mathcal{O}(10mk)$ memory. This indicates that the parent block, which was discarded by the traditional admissibility criterion (1), is compressible. The further away from the upper bound in (3) is the $\operatorname{rank}(\mathbf{Z}_{i,j}^{(\ell)})$, the more inefficient would be the result of the traditional admissibility criterion. Numerical tests indicate that the upper bound in (3) is often not tight at all; thus, augmenting the admissibility criterion in (1) as

$$\begin{gathered} \text{adm*}(\mathcal{I}_i^{(\ell)} \times \mathcal{J}_j^{(\ell)}) \ = \ \text{true} \\ \Leftrightarrow \\ \min(\operatorname{diam}(\mathcal{B}_{\mathcal{I}_i}^{(\ell)}), \operatorname{diam}(\mathcal{B}_{\mathcal{J}_j}^{(\ell)})) \le \eta \ \operatorname{dist}(\mathcal{B}_{\mathcal{I}_i}^{(\ell)}, \mathcal{B}_{\mathcal{J}_j}^{(\ell)}), \eta > 0 \\ \vee \\ \begin{pmatrix} \text{adm*}(\mathcal{I}_{i,1}^{(\ell+1)} \times \mathcal{J}_{j,1}^{(\ell+1)}) = \text{true} \ \wedge \\ \text{adm*}(\mathcal{I}_{i,1}^{(\ell+1)} \times \mathcal{J}_{j,2}^{(\ell+1)}) = \text{true} \ \wedge \\ \text{adm*}(\mathcal{I}_{i,2}^{(\ell+1)} \times \mathcal{J}_{j,1}^{(\ell+1)}) = \text{true} \ \wedge \\ \text{adm*}(\mathcal{I}_{i,2}^{(\ell+1)} \times \mathcal{J}_{j,2}^{(\ell+1)}) = \text{true} \end{pmatrix} \end{gathered} \tag{4}$$

will reduce the memory requirement and computational time for $\mathcal{H}$-LU factorization and $\mathcal{H}$-back substitution significantly. In (4), $\vee$ is the logical OR and $\wedge$ is the logical AND operation. The performance of the criterion in (4) was evaluated by constructing a fast direct solver using the algorithm in [2] and will be shown at the conference.

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