Perturbative Analysis of Electromagnetic Homogenization Near the Γ-Point in Higher Bands

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The behavior of electromagnetic waves in periodic structures near the Γ-point *in the first band* has been examined thoroughly over the years. In particular, it can be proved rigorously that (i) the dispersion relation is linear and positive (Tsukerman, JOSA B 25, 927, 2008), and (ii) if both the dispersion relation and the impedance of the medium are considered, then the only valid homogeneous description of the medium is obtained by assigning the medium a trivial magnetic permeability $\mu = 1$ (Markel & Schotland, PRE 85, 066603, 2012). For propagating waves with the wave vector $\mathbf{q} = q\hat{\mathbf{n}}$, q being a complex wave number and $\hat{\mathbf{n}}$ being a unit vector of direction, one has (for $\mu = 1$) $\mathrm{Im}\,(\mathbf{q}\cdot\mathbf{q}) = \mathrm{Im}\,q^2 = 2\,\mathrm{Re}\,q\ \mathrm{Im}\,q = (\omega/c)^2\,\mathrm{Im}\,\epsilon > 0$. This relation means that the direction of phase velocity and the direction of spatial decay of a wave always coincide. The more complicated case of waves with more general wave vectors propagating in anisotropic media is considered by Markel & Schotland (J. Opt. 12, 015104, 2010). In any case, all such phenomena are consistent with the linearity and positivity of the slope of all dispersion curves near the Γ-point in the first band. We can refer to this propagation regime as to the homogenization limit.

However, some of these results are inapplicable in the second and higher bands. Near the Γ-point, we still have $\xi \equiv |\mathbf{q}|a/\pi \ll 1$, where a is the lattice cell size. But the second parameter of the theory, namely, $\eta = \omega a/\pi c$, is not necessarily small. Note that the smallness of η was used to obtain the results outlined above. The question therefore is whether one could have a physical situation where a medium can be described as an electromagnetically-homogeneous continuum and yet $\mathrm{Im}(\mathbf{q}\cdot\mathbf{q}) < 0$. This would correspond to a medium with a negative index of refraction (Opt. Expr. **16**, 19152, 2008; J. High Energy Phys. **6**, 147, 2014). Here it is useful to clarify what exactly is meant by $|\mathbf{q}|$ in the definition of the small parameter ξ. There exist two definitions: $|\mathbf{q}| = \sqrt{\mathbf{q}^*\cdot\mathbf{q}}$ or $|\mathbf{q}| = \sqrt{|\mathbf{q}\cdot\mathbf{q}|}$. The first definition is commonly encountered in physics but we argue that it is the second definition that is relevant. This will become clear from a detailed analysis of the dispersion relations when considering a plane wave incident from a vacuum onto a metamaterial slab. We show that the isofrequency surfaces may be ellipsoidal near a higher-band Γ-point for small $|\mathbf{q}|$ where $\mathbf{q} = q\hat{\mathbf{n}}$ but that these surfaces are not necessarily ellipsoidal (in a generalized sense) for evanescent waves with $\mathbf{q} = (q_x, q_y, iq_z)$. However, waves with such $\mathbf{q}$ must be included to describe refraction from free space into the medium. In this case, the medium is not truly homogeneous, at least, it is not homogenizable with local tensors ϵ and μ.