FDTD Modeling of Temporally Dispersive Media Using a Stable Rational Function of the Z-Transform Variable

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A simple model to analyze wave propagation in dispersive media that directly maps from wide bandwidth measurements in the frequency domain to the time domain is examined. Previous work (Rappaport (2003) *IEEE EMBS Symp.*, pp. 3789—3782) using a rational function of conductivity, in terms of the Z-transform variable, $Z = \exp[j\omega\Delta t]$, is extended to provide a straightforward, user-determined formulation, particularly applicable to FDTD. The current formulation incorporates an additional parameter, but avoids the inaccuracies of averaging the current over two time steps, uses the measured value of dielectric constant in the middle of the frequency band as the average value, and is straightforward to test for stability. This formulation avoids memory-intensive convolution operations and is at least as accurate as Debye models. Tests on various measured biological tissues and soils indicate waves propagated using FDTD maintain the same velocity and attenuation characteristics, (within 5%) as predicted in the frequency domain.

In the FDTD formulation, with electric field sampled at integer time steps E^n, and magnetic field sampled at half-integer steps $H^{n+1/2}$, Ampere's Law presents a difficulty with the current term, which is computed using electric field but which must be available at the magnetic field time instant. The half time step current can be accounted for by multiplying the Z-transformed conductivity by $Z^{1/2}$. The single-pole rational function representation for conductivity is given by:

$$Z^{-\frac{1}{2}}\sigma(Z) = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + b_3 Z^{-3}}{1 + a_1 Z^{-1}}$$

With this choice, the entire right hand side of Ampere's law remains a rational function of integer powers of Z, and thus it can be readily converted to finite difference form.

The additional term $b_3 Z^3$ is necessary to ensure three point fitting, with proper curvature, of the conductivity function to measured data. The real part of conductivity,

$$\operatorname{Re}\{\sigma(Z)\} = \frac{[b_0 + b_1 + a_1(b_1 + b_2)]\cos\frac{\omega\Delta t}{2} + (b_2 + a_1(b_0 + b_3))]\cos\frac{3\omega\Delta t}{2} + b_3\cos\frac{5\omega\Delta t}{2}}{1 + 2a_1\cos\omega\Delta t + a_1^2}$$

is determined by fitting the five parameters a_1, b_0, b_1, b_2, and b_3 to measured conductivity data across the bandwidth of interest. The parameter a_1 is adjusted to satisfy special von Neumann stability conditions requiring that all zeros of the stability equation be within the unit circle for a particular grid spacing interval.

This work is supported by CenSSIS, the Center for Subsurface Sensing and Imaging Systems under the ERC Program of the NSF (Award number EEC-9986821).