Frequency- and Time-Domain Scattering from DNG Slabs

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Introduction

The main purpose of this paper is to explain and resolve a number of the peculiarities (such as perfectly reproduced source fields as well as divergent fields to the right of the slab) exhibited by the solution to a single-frequency sinusoidal source illuminating a lossless magnetodielectric slab with relative permittivity and permeability equal to negative one [1]–[6]. This is accomplished by showing that the solution for a time-domain sinusoidal source, that turns on at $t = 0$ and illuminates a lossless slab with the slowest possible frequency variation in permittivity and permeability allowed by causality and energy conservation, has finite fields everywhere and imperfectly resolved source fields for all finite time t that the source is turned on. To do this, we begin with the frequency-domain solution.

Frequency-Domain Solution

The plane-wave solution [7]–[10] for the x component of the electric field $E_x(x,z)$ of a transverse electric (TE) ($E_y = 0$, $H_y \neq 0$) time-harmonic ($e^{-i\omega t}$, $\omega > 0$) line source with no variation in the y direction located a distance d in front of a slab (infinite in the x and y directions and normal to z) with width L, complex permittivity $\epsilon = \epsilon' + i\epsilon''$, and complex permeability $\mu = \mu' + i\mu''$ is given by

$$E_x(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ihx}dh \begin{cases} T_0(h)e^{i\gamma_0 z} + R_0(h)e^{-i\gamma_0 z} & , \quad 0 < z \leq d \\ T_s(h)e^{i\gamma z} + R_s(h)e^{-i\gamma z} & , \quad d \leq z \leq d+L \\ T(h)e^{i\gamma_0 z} & , \quad d+L \leq z \end{cases} \tag{1}$$

$$\gamma_0 = (k_0^2 - h^2)^{\frac{1}{2}}\,, \quad k_0^2 = \omega^2\mu_0\epsilon_0\,, \qquad \gamma = (k^2 - h^2)^{\frac{1}{2}}\,, \quad k^2 = \omega^2\mu\epsilon \tag{2}$$

where ϵ_0 and μ_0 are the permittivity and permeability of the free space in which the slab is assumed located. For passive materials $\epsilon'' \geq 0$ and $\mu'' \geq 0$. The square root in the definition in (2) of γ_0 is chosen positive real or positive imaginary depending upon whether $h^2 < k_0^2$ or $h^2 > k_0^2$, respectively. The sign of the square root in the definition in (2) of γ is chosen to keep the imaginary part of γ positive. If k^2 is real and $h^2 < k^2$, then γ is real and the sign of γ is found by inserting a small loss, choosing the imaginary part of γ positive, and letting the loss approach zero. This leads to a positive [negative] real γ if ϵ and μ are both positive [negative] real.

The plane-wave spectrum $T_0(h)$ of the fields to the right ($z > 0$) of the line source (incident fields) is assumed given. For a two-dimensional y directed magnetic line current (magnetization), $T_0(h)$ is independent of h and can be written as $T_0(h) = E_0/k_0$, where E_0 is a constant with electric field units and the constant k_0 is inserted into the denominator to ensure the proper dimension of $E_x(x,z)$. The reflected spectrum $R_0(h)$ to the left of the slab, the transmitted and reflected spectra, $T_s(h)$ and $R_s(h)$, within the slab, and the transmitted spectrum $T(h)$ to the right of the slab are obtained by equating the tangential components of the electric and magnetic fields across the interfaces of the slab at $z = d$ and $z = d + L$. For example, $T(h) = T_0(h)\mathcal{T}_{\rm TE}(h)$, where the TE transmission coefficient is given as

$$\mathcal{T}_{\rm TE}(h) = \frac{4e^{-i\gamma_0 L}}{\left(2 + \frac{\epsilon\gamma_0}{\epsilon_0\gamma} + \frac{\epsilon_0\gamma}{\epsilon\gamma_0}\right) e^{-i\gamma L} + \left(2 - \frac{\epsilon\gamma_0}{\epsilon_0\gamma} - \frac{\epsilon_0\gamma}{\epsilon\gamma_0}\right) e^{i\gamma L}} \,. \tag{3}$$

In the case of the "perfectly focusing" slab [1], [2], $\epsilon/\epsilon_0 = \mu/\mu_0 = -1$ at some frequency ω_0 and we have $\gamma = -\gamma_0$ if $h^2 < k^2 = k_0^2 = \omega_0^2\mu_0\epsilon_0$ and $\gamma = \gamma_0$ if $h^2 > k_0^2$. Then $\mathcal{T}_{\rm TE}(h) = e^{-i\gamma_0 2L}$ and this leads to

$$E_x(x,z) = \begin{cases} E_x^{\rm inc}(x,z) & , \quad 0 < z \le d \\ E_x^{\rm inc}(x, 2d - z) & , \quad d \le z < 2d \\ \text{infinite divergent field} & , \quad 2d < z < 2L \\ E_x^{\rm inc}(x, z - 2L) & , \quad 2L < z \end{cases} \tag{4}$$

where the incident field to the right of the source in free space is given by

$$E_x^{\rm inc}(x,z) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} T_0(h) e^{i(hx + \gamma_0 z)} dh \,, \quad z > 0 \,. \tag{5}$$

Herein out it is assumed that $d < L$. In other words, the field to the right of the source and to the left of the slab is just the incident field of the source in free space. The field to the right of the front face of the slab and to the left of $z = 2d$ is the image of the source field. The field between $z = 2d$ and $z = 2L$ diverges to infinite values. Remarkably, the field in free space to the right of $z = 2L$ is just the incident field translated to the right a distance equal to twice the width of the slab. This perfect replication of the free-space source fields for $z > 0$ in the free-space region $z > 2L$ to the right of the slab is sometimes referred to as "perfect focusing" [1], [2].

The solution in (4) is so unusual that it warrants further investigation. It is a solution that assumes the loss in the slab (and the surrounding space) is exactly zero. It may be physically more appealing to insert a small loss into the slab and determine the solution for the fields as the loss approaches zero. For simplicity, let the relative loss at the frequency ω_0 in $\mu(\omega_0)$ and $\epsilon(\omega_0)$ of the slab be equal and denoted by $\delta''(\omega_0) = \mu''(\omega_0)/\mu_0 = \epsilon''(\omega_0)/\epsilon_0 > 0$. Then $\epsilon\gamma_0/(\epsilon_0\gamma) = -1 + i\delta''(1 + k_0^2/|\gamma_0|^2) + O[(\delta'')^2] \approx -1 + i\delta'' + O[(\delta'')^2]$ for the evanescent spectrum if terms in $k_0^2/|\gamma_0|^2$ are neglected compared to unity and we find that $\mathcal{T}_{\rm TE}(h)$ in (3) for $\delta'' \ll 1$ can be approximated by

$$\mathcal{T}_{\rm TE}(h) \approx \frac{e^{|\gamma_0|L}}{\frac{\delta''^2}{4}\, e^{|\gamma_0|L} + e^{-|\gamma_0|L}} \,, \quad h^2 > k_0^2 \,. \tag{6}$$

This expression reveals that $\mathcal{T}_{\rm TE}(h)e^{-|\gamma_0|z}$, which comprises the evanescent integrand in the region $z \ge d + L$, rapidly decreases toward zero as $|h|$ grows larger than

$$H_\delta \approx \sqrt{\left(\frac{2}{z}\ln\delta''\right)^2 + k_0^2} \,. \tag{7}$$

We then find as $\delta'' \to 0$

$$E_x^{\delta''\to 0}(x,z) = \begin{cases} \text{bounded field} & , \quad 0 < z < d - (L - d) \\ \text{infinite divergent field} & , \quad d - (L-d) < z < d + (L - d) \\ \text{bounded field} & , \quad d + (L - d) < z < 2d \\ \text{infinite divergent field} & , \quad 2d < z < 2L \\ E_x^{\rm inc}(x, z - 2L) & , \quad 2L < z \end{cases} \tag{8}$$

where it is assumed in (8) that $L/2 < d < L$. The fields in (8) conform to those obtained in [11].

Comparing (8) and (4) reveals that the fields in the region $0 < z < 2d$ differ depending upon whether the loss δ'' in the slab is made to approach zero before [to get(4)] or after [to get(8)] the limit of the integration of the evanescent spectrum is allowed to approach infinity. However, in the important free-space region to the right of the slab ($z > d + L$), the nature of the fields is independent of the order in which the limit of the loss (approaching zero) and the limit of the integration of the evanescent spectrum (approaching infinity) is taken. In either case, "perfect focusing" of the source fields for $z > 0$ is attained in the free-space region $z > 2L$ to the right of the slab. Also, in either case, the field diverges to infinite values in the free-space region $d + L < z < 2L$ that lies to the right of the slab. (Note that an electric field component in a free-space region need not be an analytic function of the spatial coordinates throughout that region if the field is allowed to diverge to infinite values in part of that region.)

The "resolution enhancement" R_e, defined as the ratio of the resolution with H_δ to the resolution with the propagating waves alone ($H_\delta = k_0$), is given just to the right of $z = 2L$ by

$$R_e = \frac{H_\delta}{k_0} \approx \sqrt{\left(\frac{\lambda_0}{2\pi L}\ln\delta''\right)^2 + 1}\,. \tag{9}$$

For $\lambda_0 \ln\delta''/(2\pi L) \gg 1$, (9) reduces to $R_e \approx \lambda_0 \ln\delta''/(2\pi L)$ [12]. The increasingly large fields in the region $d + L < z < 2L$ can be found asymptotically as

$$E_x^\delta \overset{\delta''\to 0}{\approx} \frac{E_0}{\pi k_0\sqrt{x^2 + (2L - z)^2}} \frac{1}{(\delta'')^{2(2L/z-1)}} \cos\left(\frac{2x}{z}\ln\delta'' + \tan^{-1}\frac{x}{2L - z}\right). \tag{10}$$

The asymptotic $(\delta'')^{2(2L/z-1)}$ decay in (10) is a more accurate than the approximate decay of $(\delta'')^{2-z/L}$ determined for the quasistatic fields in [11].

Time-Domain Solution

Let T_ω denote the frequency-domain spectrum of the same line source used above but now excited by a sinusoidal wave, $\cos(\omega_0 t)$, that turns on at the initial time $t = 0$ and stays on through the present time t. Then the real-valued time-domain electric field $E_x(x, z, t)$ can be expressed in the region $z \geq d + L$ as [10, sec. 5.3]

$$E_x(x, z, t) = \mathrm{Re}\,[\overset{+}{E}_x(x, z, t)] \tag{11}$$

with

$$\overset{+}{E}_x(x, z, t) = \frac{1}{\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} T_\omega T_0(h)\mathcal{T}_{\mathrm{TE}}(h)e^{i(hx+\gamma_0 z-\omega t)}dh\,d\omega\,, \quad z \geq d + L\,. \tag{12}$$

For large t, the dominant contribution in (12) will come from the ω integration near ω_0. Therefore, the next step toward evaluating the integral in (12) analytically is to expand the transmission coefficient $\mathcal{T}_{\mathrm{TE}}(h)$ in a power series about $\omega = \omega_0$. Because our primary interest lies in evaluating (12) for a lossless slab with relative permittivity and permeability equal to -1 at the chosen frequency $\omega = \omega_0$, a power series expansion of $\epsilon(\omega)/\epsilon_0$ and $\mu(\omega)/\mu_0$, which are needed in $\mathcal{T}_{\mathrm{TE}}(h)$, is

$$\frac{\epsilon(\omega)}{\epsilon_0} = -1 + \frac{1}{\epsilon_0}\frac{d\epsilon}{d\omega}(\omega_0)(\omega - \omega_0) + O\left[(\omega - \omega_0)^2\right] \tag{13}$$

and similarly for $\mu(\omega)$. For lossless materials, both causality [13, sec. 84] and energy conservation [14, app. B] require that these coefficients have the lower bounds

$$\frac{1}{\epsilon_0}\frac{d\epsilon}{d\omega}(\omega_0) \geq \frac{4}{\omega_0}\,, \qquad \frac{1}{\mu_0}\frac{d\mu}{d\omega}(\omega_0) \geq \frac{4}{\omega_0}\,. \tag{14}$$

Consequently, the $\frac{d\epsilon}{d\omega}(\omega_0)/\epsilon_0$ and $\frac{d\mu}{d\omega}(\omega_0)/\mu_0$ that vary the least rapidly near $\omega = \omega_0$ (and thus will reproduce the source fields most closely in the region $z > 2L$) are

$$\kappa(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} = \frac{\mu(\omega)}{\mu_0} = -1 + \frac{4}{\omega_0}(\omega - \omega_0) + O\left[(\omega - \omega_0)^2\right]\,. \tag{15}$$

With (15) inserted into the expression for $\mathcal{T}_{\rm TE}$ in (3), one finds that for the propagating waves ($h^2 < k_0^2$) it can be approximated by $\mathcal{T}_{\rm TE}(h) \approx e^{-i\gamma_0 2L}$, and for the evanescent waves ($h^2 > k_0^2$), the quantity $\kappa(\omega)\gamma_0/\gamma = -1 + 4(1 + k_0^2/|\gamma_0|^2)(\omega - \omega_0)/\omega_0 + O\left[(\omega-\omega_0)^2\right] \approx -1 + 4(\omega-\omega_0)/\omega_0 + O\left[(\omega-\omega_0)^2\right]$ if terms in $k_0^2/|\gamma_0|^2$ are neglected compared to unity and we find

$$\mathcal{T}_{\rm TE}(h) \approx \frac{e^{|\gamma_0|L}}{-\frac{4}{\omega_0^2}(\omega-\omega_0)^2\, e^{|\gamma_0|L} + e^{-|\gamma_0|L}}\,, \quad h^2 > k_0^2\,. \tag{16}$$

Substitution of $\mathcal{T}_{\rm TE}$ into (12), and an asymptotic integration over ω, leads to

$$\overset{+}{E}_x(x,z,t) \approx \frac{e^{-i\omega_0 t}}{2\pi} \int_{-H_t}^{+H_t} T_0(h) e^{i[hx + \gamma_0(z-2L)]} dh\,, \quad z \geq d + L \tag{17}$$

with

$$H_t \approx \sqrt{\left[\frac{2}{z}\ln(f_0 t)\right]^2 + k_0^2}\,, \quad f_0 = \frac{\omega_0}{2\pi}\,. \tag{18}$$

The resolution enhancement as a function of the on-time t is

$$R_e = \frac{H_t}{k_0} \approx \sqrt{\left[\frac{\lambda_0}{2\pi L}\ln\left(f_0 t\right)\right]^2 + 1} \tag{19}$$

and an asymptotic expression for $\overset{+}{E}_x(x,z,t)$ in $d + L < z < 2L$ as $t \to \infty$ is

$$\overset{+}{E}_x \overset{t\to\infty}{\approx} \frac{E_0\, e^{-i\omega_0 t}}{\pi k_0 \sqrt{x^2 + (2L-z)^2}}\, \tau^{2(2L/z-1)} \cos\left(\frac{2x}{z}\ln\tau - \tan^{-1}\frac{x}{2L-z}\right) \tag{20}$$

with $\tau = f_0 t$, confirming the fields diverge as $t \to \infty$ in the region $d + L < z < 2L$.

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