**Transformation Procedure of Unit Vectors to Facilitate Evaluation of Vector Potentials**

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## I. Introduction

A simple method is presented which details the process by which *unit vectors* in a source region are represented in terms of unit vectors of a field point. Since everything is done from the standpoint of the source unit vectors, this procedure differs slightly from equivalent treatments that transform distributions of source currents [1 – 3]. The objective of this paper is to present this method in a systematic fashion which easily translates to a number of coordinate systems. Using the method of the gradient and a high-level language such as Mathematica, these conversions can be done with great ease. This approach finds its greatest value when considering coordinate systems that are not used widely used.

## II. Background

The evaluation of an integral containing a Green's function and current distribution is fundamental step when determining the radiation patterns for many types of antennas. For example, the integral for the vector potential $\overline{A}$ (free space), where the integrand contains the free-space Green's Function and a distribution of current, requires integration over a source region. Setting up the integrand is straightforward when the distributions are expressed in Cartesian coordinates. However, care must be taken if the integration is to employ other coordinate systems. A good example of the complexities that arise is seen when evaluating $\overline{A}$ for a loop of current [1]. This paper demonstrates that the integrals can be properly formulated through a transformation of the source unit vectors.

A specific integral of interest (for the vector potential $\overline{A}$) is evaluated in the far field using the following form [1]:

$$\overline{A}(\overline{r}) = \frac{\mu_o}{4\pi}\frac{e^{-jkr}}{r}\int_{v'} \overline{J}'(\overline{r}')\, e^{jk\hat{r}\cdot\overline{r}'}\, dv' \tag{1}$$

The evaluation of $\overline{A}$ is done over a coordinate system suitable for the distribution of source current (electric or magnetic). The vector $\overline{r}'$ (defined as a vector from the origin of the coordinate system to the source) is frequently expressed in Cartesian, Cylindrical, or Spherical coordinates; although, other coordinate systems may also be employed depending on the nature of the current distribution. In these three coordinate systems, $\overline{r}'$ is expressed as:

$$\overline{r}' = \hat{x}'x' + \hat{y}'y' + \hat{z}'z' \tag{2}$$

$$\overline{r}' = \hat{\rho}'\rho' + \hat{z}'z' \tag{3}$$

$$\overline{r}' = \hat{r}'r' \tag{4}$$

where the unit vectors (e.g., $(\hat{x}', \hat{y}', \hat{z}')$) have been annotated with prime notation emphasizing they are describing the source region. The vector denoting the *field position* is denoted as: $\bar{r} = \hat{r}\, r$ and $r = |\bar{r}|$. Proper transformations remove the prime notation from the unit vectors.

**III. Transformation of the Unit Vectors**

The underlying problem is to represent the *prime* unit vectors (describing the source region) in terms of the *unprimed* unit vectors (representing the field point). The fact that these unit vectors in Cartesian coordinates translate directly from primed to unprimed notation can then be exploited, that is:

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{5}$$

Note that this paper will impose the restriction that the source (antenna) and field coordinate systems are located at the same point.

While there is a direct projection of unit vectors in prime and unprimed notation for Cartesian coordinates (i.e., $\hat{x}' \cdot \hat{x} = 1$), unit vectors in other coordinate systems (or in two different coordinate systems) *do not* project directly on each other. For example, when using spherical coordinates for both the source and field $\hat{r}' \cdot \hat{r} \neq 1$ (except for the special case when $\phi = \phi'$ and $\theta = \theta'$), that is:

$$\begin{bmatrix} \hat{r}' \\ \hat{\theta}' \\ \hat{\phi}' \end{bmatrix} \neq \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{6}$$

Equation 5 can be used to determine the relationship between two different coordinate systems. For example, unit vectors in spherical coordinates can be decomposed into equivalent Cartesian coordinate unit vectors by:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} Sin(\theta)Cos(\phi) & Cos(\theta)Cos(\phi) & -Sin(\phi) \\ Sin(\theta)Sin(\phi) & Cos(\theta)Sin(\phi) & Cos(\phi) \\ Cos(\theta) & -Sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{7}$$

The source (primed) unit vectors in Cartesian coordinates are easily obtained substituting Equation 5 into 7.

The problem of representing the primed cylindrical coordinate unit vectors in terms of spherical coordinate unit vectors is now addressed. The primed unit vector representation is given by:

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} Cos(\phi') & -Sin(\phi') & 0 \\ Sin(\phi') & Cos(\phi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho}' \\ \hat{\phi}' \\ \hat{z}' \end{bmatrix} \tag{8}$$

By Equation 5, it follows that Equations 7 and 8 may be equated, that is:

$$\begin{bmatrix} Cos(\phi') & -Sin(\phi') & 0 \\ Sin(\phi') & Cos(\phi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho}' \\ \hat{\phi}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} Sin(\theta)Cos(\phi) & Cos(\theta)Cos(\phi) & -Sin(\phi) \\ Sin(\theta)Sin(\phi) & Cos(\theta)Sin(\phi) & Cos(\phi) \\ Cos(\theta) & -Sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{9}$$

Solving for the primed unit vectors gives:

$$\begin{bmatrix} \hat{\rho}' \\ \hat{\phi}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} Sin(\theta)Cos(\phi-\phi') & Cos(\theta)Cos(\phi-\phi') & -Sin(\phi-\phi') \\ Sin(\theta)Sin(\phi-\phi') & Cos(\theta)Sin(\phi-\phi') & Cos(\phi-\phi') \\ Cos(\theta) & -Sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{10}$$

One can follow this same procedure for a source expressible in primed spherical coordinate unit vectors, the transformation to unprimed spherical coordinate unit vectors is:

$$\begin{bmatrix} \hat{r}' \\ \hat{\theta}' \\ \hat{\phi}' \end{bmatrix} = \begin{bmatrix} \begin{matrix} Cos(\theta)Cos(\theta')+ \\ Sin(\theta)Sin(\theta')Cos(\phi-\phi') \end{matrix} & \begin{matrix} -Sin(\theta)Cos(\theta')+ \\ Cos(\theta)Sin(\theta')Cos(\phi-\phi') \end{matrix} & -Sin(\theta')Sin(\phi-\phi') \\ \begin{matrix} -Cos(\theta)Sin(\theta')+ \\ Sin(\theta)Cos(\theta')Cos(\phi-\phi') \end{matrix} & \begin{matrix} Sin(\theta)Sin(\theta')+ \\ Cos(\theta)Cos(\theta')Cos(\phi-\phi') \end{matrix} & -Cos(\theta')Sin(\phi-\phi') \\ Sin(\theta)Sin(\phi-\phi') & Cos(\theta)Sin(\phi-\phi') & Cos(\phi-\phi') \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{11}$$

Equations 10 and 11 contain terms that enter into the integration process.

**III. Other Coordinate Systems**

The method of solution used in this paper can be extended to any coordinate system of interest and is briefly outlined. The "method of the gradient" [4] facilitated the work and was employed extensively. For example, Paraboloidal coordinates (unit vectors of $(\hat{u}', \hat{v}', \hat{\phi}')$) have scalar lengths $(x', y', z')$ given by [5, pp. 34]:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} u'v'Cos(\phi') \\ u'v'Sin(\phi') \\ \left(u'^2 + v'^2\right)/2 \end{bmatrix} \tag{12}$$

Taking the gradient (Cartesian) of the left and the gradient of the right (Paraboloidal) of Equation 12 gives:

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \frac{v'Cos(\phi')}{\sqrt{u'^2+v'^2}} & \frac{u'Cos(\phi')}{\sqrt{u'^2+v'^2}} & -Sin(\phi') \\ \frac{v'Sin(\phi')}{\sqrt{u'^2+v'^2}} & \frac{u'Sin(\phi')}{\sqrt{u'^2+v'^2}} & Cos(\phi') \\ \frac{u'}{\sqrt{u'^2+v'^2}} & \frac{v'}{\sqrt{u'^2+v'^2}} & 0 \end{bmatrix} \begin{bmatrix} \hat{u}' \\ \hat{v}' \\ \hat{\phi}' \end{bmatrix} \tag{13}$$

As done previously, the right-hand sides of Equations 7 and 13 are equated and solved for the primed unit vectors.

$$\begin{bmatrix} \hat{u}' \\ \hat{v}' \\ \hat{\phi}' \end{bmatrix} = \begin{bmatrix} \dfrac{u' Cos(\theta) - v' Cos(\phi-\phi') Sin(\theta)}{\Delta \sqrt{u'^2 + v'^2}} & \dfrac{u' Sin(\theta) + v' Cos(\phi-\phi') Cos(\theta)}{-\Delta \sqrt{u'^2 + v'^2}} & \dfrac{Sin(\phi-\phi') v'}{\Delta \sqrt{u'^2 + v'^2}} \\ \dfrac{u' Cos(\phi-\phi') Sin(\theta) - v' Cos(\theta)}{\Delta \sqrt{u'^2 + v'^2}} & \dfrac{u' Cos(\theta) Cos(\phi-\phi') + v' Sin(\theta)}{\Delta \sqrt{u'^2 + v'^2}} & \dfrac{Sin(\phi-\phi') u'}{-\Delta \sqrt{u'^2 + v'^2}} \\ Sin(\theta) Sin(\phi-\phi') & Cos(\theta) Sin(\phi-\phi') & Cos(\phi-\phi') \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix}$$

$$\Delta = \frac{u'^2 - v'^2}{u'^2 + v'^2} \tag{14}$$

The method is readily used for the coordinate systems discussed in [5] such as Elliptic-cylinder, Prolate-spheroidal, Oblate-spheroidal, etc.

## IV. Conclusion

While equivalent methodologies are well known and have been employed over the years to evaluate problems such as the loop of current, it is believed that the approach outlined here streamlines the formulation of the integrand and made the analysis of such problems more systematic. Since the integrals for $\overline{A}$ (and $\overline{F}$ ) have been treated extensively in the Cartesian, Cylindrical, and Spherical coordinate systems, the approach advocated here merely outlines an alternate way to devise the integrals. The primary advantage of this method is its systematic process of unit vector transformation, which was extendable to any desired coordinate system.

## References


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[5] P. Moon and D. E. Spencer, *Field Theory Handbook*, New York: Springer-Verlag, 2nd edition (corrected 3rd printing), 1988.