The Complete Free Space Time Domain Green's Function and Propagator for Maxwell's Equations

Robert Nevels
Department of Electrical Engineering
Texas A&M University
College Station, Texas 77843-3128
nevels@ee.tamu.edu

Introduction

The propagator is a mathematical expression which, when convolved with any given present time field, evolves that field through a predetermined time increment [1-4]. When the field is entirely causal, the free space propagator and free space Green's function have a simple mathematical relationship. In this paper a method for finding the full wave time domain propagator for the electromagnetic field is presented. Starting with Maxwell's differential equations in tensor form, a state variable approach is used to derive expressions for the propagator in three dimensions. It is shown that the properties of the propagator, which satisfies a homogeneous hyperbolic matrix equation, and the Green's function, which satisfies an inhomogeneous equation with the same operator, can be used to determine their mathematical relationship.

Formulation

In a source free homogeneous region the time domain Maxwell curl equations in terms of the electric and magnetic field intensities, $\mathbf{E}$ and $\mathbf{H}$, are

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

where the permeability and permittivity are respectively μ and ε. Eqns. (1) and (2) can be cast in the general matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \bar{\mathbf{S}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{3}$$

where

0-7803-7846-6/03/$17.00 ©2003 IEEE

$$\overline{\mathbf{S}} = \begin{pmatrix} \overline{\mathbf{0}} & \varepsilon^{-1}\overline{\mathbf{R}} \\ -\mu^{-1}\overline{\mathbf{R}} & \overline{\mathbf{0}} \end{pmatrix} \tag{4}$$

$\overline{\mathbf{0}}$ is a 3×3 null matrix and $\overline{\mathbf{R}}$ is the operator matrix

$$\overline{\mathbf{R}} = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & 0 & -\dfrac{\partial}{\partial x} \\ -\dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \end{pmatrix} \tag{5}$$

Referring to (3) the propagator $\overline{\mathbf{K}}$ is found by solving

$$\frac{\partial \overline{\mathbf{K}}}{\partial t} - \overline{\mathbf{S}}\overline{\mathbf{K}} = \overline{\mathbf{0}} \tag{6}$$

subject to the conditions

$$\lim_{t \to 0} \overline{\mathbf{K}} = \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') \tag{7}$$

Eqn. (6) can be solved to find the components of the propagator matrix

$$\overline{\mathbf{K}} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & 0 & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & 0 \\ 0 & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & 0 & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & 0 & K_{64} & K_{65} & K_{66} \end{bmatrix} \tag{8}$$

Standard Fourier transform techniques yield a solution to (6) in the form

$$\overline{\mathbf{K}} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{\overline{\mathbf{S}}\tau} e^{j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} d\mathbf{k} \tag{9}$$

After tedious mathematical effort, (9) yields the elements of (8). As an example of the general form of these elements,

$$\begin{aligned} K_{11} = {} & \left[\frac{\delta(R - v\tau) - 2\delta(R)}{4\pi R^2} + \frac{U(v\tau - |R|)}{4\pi R^3}\right]\left[1 - \frac{3(x - x')^2}{R^2}\right] \\ & + \left[\frac{\delta'(R - v\tau) - 2\delta'(R)}{4\pi R}\right]\left(\frac{x - x'}{R}\right)^2 - \frac{\delta'(R - v\tau)}{4\pi R} \end{aligned} \tag{10}$$

where $R=\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}$, $\tau=t-t_0$, v is the velocity of the wave in the homogeneous medium, δ,δ' are the Dirichlet delta function and its derivative and the unit step is defined by

$$U(v\tau-|R|)=\begin{cases}1, |R|<v\tau\\ 0, |R|>v\tau\end{cases}.$$

Once $\overline{\mathbf{K}}$ is known, the field everywhere at a time t can be found from the previous time (t_0) field by performing the convolution operation:

$$\mathbf{F}(\mathbf{r},t)=\int_{-\infty}^{\infty}\mathbf{F}_o(\mathbf{r}')\overline{\mathbf{K}}(\mathbf{r},t\,|\,\mathbf{r}',t_o)d\mathbf{r}',\quad \mathbf{F}=[\mathbf{E}\quad \mathbf{H}]^T \tag{11}$$

It is interesting that the elements of the propagator matrix $\overline{\mathbf{K}}$, all of which are similar to K_{11}, are composed of simple analytical expressions. According to Eqn. (11), the various propagator terms form the final field by the accumulated effect of the initial field and its derivative on the spherical causal boundary, due to the $\delta(R-v\tau)$ and $\delta'(R-v\tau)$ terms, the initial field and its derivative at the center of the causal sphere, due to the $\delta(R)$ and $\delta'(R)$ terms, and the initial field everywhere between the center and boundary of the causal sphere, due to the $U(v\tau-|R|)$ terms. This is somewhat puzzling in that one would expect that only the initial field on the spherical causal boundary and traveling toward the center should have an effect on the field at the center of the causal sphere at the end of the causal time increment.

Although Eqn. (11) appears to be the tensor form of Huygens principle in the time domain, the propagator $\overline{\mathbf{K}}$ satisfies the homogeneous equation

$$\frac{\partial\overline{\mathbf{K}}}{\partial t}-\overline{\mathbf{S}}\overline{\mathbf{K}}=0, \tag{12}$$

whereas the Green's function, which is the kernel generally associated with Huygens principle, satisfies

$$\frac{\partial\overline{\mathbf{G}}}{\partial t}-\overline{\mathbf{S}}\overline{\mathbf{G}}=\overline{\boldsymbol{\delta}}(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{13}$$

$$\overline{\mathbf{G}}=0\qquad \text{for}\qquad t-t'<0 \tag{14}$$

However, it can be shown that there is a simple relationship between the propagator and Green's function. First observe that

$$\lim_{t \to t'^{+}} \overline{\mathbf{G}}(\mathbf{r},t\,|\,\mathbf{r}',t') = \lim_{t \to t'^{+}} \overline{\mathbf{K}}(\mathbf{r},t\,|\,\mathbf{r}',t') = \overline{\mathbf{K}}(\mathbf{r},t'\,|\,\mathbf{r}',t') = \overline{\boldsymbol{\delta}}(\mathbf{r}-\mathbf{r}') \tag{15}$$

It is therefore postulated that the relationship between the time domain propagator and Green's functions for Maxwell's equations is

$$\overline{\mathbf{G}}(\mathbf{r},t\,|\,\mathbf{r}',t') = U(t-t')\overline{\mathbf{K}}(\mathbf{r},t\,|\,\mathbf{r}',t') \tag{16}$$

That this is indeed the case can easily be shown by substituting (16) into (13) and appealing to (12), (14) and (15).

Conclusions

The free space time domain propagator for Maxwell's equations has been found and from this the time domain Green's function has been derived. It is shown that the two are the same function after the initial time $t > t'$. It is also shown that the propagator/Green's function matrix has terms that are surprisingly simple. Possible applications of this result are as an alternative scattering formulation to time domain integral equations, which typically are based on potential functions [5]. Also, the generality of this analytical result suggests that one might be able to use the propagator expressions to accurately determine the error incurred in a variety of time domain numerical methods.

References

[1] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, New York: McGraw Hill, 1965.

[2] V. S. Bushlaev, "Continuum Integrals and the Asymptotic Behavior of the Solutions of Parabolic Equations as $t \to 0$. Applications to Diffraction," in *Topics in Mathematical Physics*, vol. 2. Edited by M. Sh. Birman, Consultants Bureau, New York, 1968.

[3] J. B. Keller and D. W. McLaughlin, "The Feynman Integral," *The Amer. Math. Monthly*, vol. 82, No. 5, pp. 451-465, 1975.

[4] R.D. Nevels, J.A. Miller, and R.M. Miller, 'A Path Integral Time Domain Method for Electromagnetic Scattering,' *IEEE Transactions on Antennas and Propagation*, vol. 48, no. 4, pp. 565-573, April 2000.

[5] S. M. Rao, '*Time Domain Electromagnetics*', San Diego, Academic Press, 1999.