Volume Integral Equations for Permeable Structures

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I. Introduction

Approaches involving integral equation methods require substantially more computational resources due to their global nature. This is further exacerbated for volume integral equations (VIEs) which traditionally require six scalar unknowns per volume location for non-trivial permittivity and permeability. Excessive sampling requirements [1] for high contrast materials add to the computational requirements. Thus, numerical simulations have so far been focused on purely dielectric bodies ($\mu_r = 1$). Initial integral equation implementations were done by Richmond [2] for two-dimensional scatterers and later by Livesay and Chen [3] and Schaubert et. al. [4] for three dimensions (see also Peterson [5] and Graglia et. al. [6]). Use of fast integral methods can remove the computational bottleneck associated with VIEs. However, this paper is focused on the derivation of a VIE which involves only a single field quantity as the unknown and does not contain any surface integrals.

In this paper we present a rigorous derivation of a VIE to model inhomogeneous targets with non-trivial permittivity and permeability using a single vector field unknown (3 scalar unknowns per location) are proposed here. As opposed to the derivation in [7], our VIE does not involve any differentiation of the permeability within the inhomogeneous region (see Figure 1 (a)). In addition, issues related to the possible presence of surface integral over the surface enclosing a discontinuous dielectric are resolved. However, the derived VIE does involve differentiation of the unknown field $\mathbf{E}$ or $\mathbf{H}$ in much the same way as in standard finite element formulations [9]. A method of moments (MoM) solution using conformal volumetric elements will also be presented. Validation data using this approach are given by comparison with results from the well-established finite element-boundary integral (FE-BI) method using the same volumetric elements as those in [8]. We remark, this paper does not involve any claims on the efficiency of the VIE and FE-BI methods. The entire focus of the paper is on the correctness and rigor of the VIEs.

II. Volume Integral Equation Derivation

We begin by referring to Fig. 1 (a) and introduce the dyadic $\overline{\mathbf{Q}}(\mathbf{r},\mathbf{r}')$,

$$\overline{\mathbf{Q}}(\mathbf{r},\mathbf{r}') = \mathbf{E}(\mathbf{r}') \times \left[\nabla' \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right] + i\omega\mu_0\mathbf{H}(\mathbf{r}') \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'), \tag{1}$$

with $\mathbf{r}$ and $\mathbf{r}'$ referring to the usual observation and source position vectors. Also, the dyadic $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ refers to the free space Green's function and $\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon_0\overline{\epsilon}_r(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$ for $\mathbf{r} \in v$, whereas $\nabla \times \mathbf{H}(\mathbf{r}) =$

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$-i\omega\epsilon_0\overline{\mathbf{I}}\cdot\mathbf{E}(\mathbf{r})+\mathbf{J}(\mathbf{r})$ for $\mathbf{r}\in v_0$ (the dual expressions also hold). We remark that the source current $\mathbf{J}(\mathbf{r})$ is non-zero in v_s which is contained within v_0. In proceeding with the derivation of the VIE, the usual approach is to take the divergence of $\overline{\mathbf{Q}}$ and then employ the divergence theorem. We consider two cases: (a) $\mathbf{r}'\in v_0$ and (b) $\mathbf{r}'\in v$. For $\mathbf{r}'\in v_0$ we have,

$$\begin{aligned}\nabla'\cdot\overline{\mathbf{Q}}(\mathbf{r},\mathbf{r}') = &+ \quad i\omega\mu_0\mathbf{H}(\mathbf{r}')\cdot\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]-k_0^2\mathbf{E}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \\ &- \quad i\omega\mu_0\mathbf{H}(\mathbf{r}')\cdot\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]+k_0^2\mathbf{E}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \\ &- \quad \mathbf{E}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')+i\omega\mu_0\mathbf{J}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \\ = &- \quad \mathbf{E}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')+i\omega\mu_0\mathbf{J}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'),\end{aligned} \tag{2}$$

and for $\mathbf{r}'\in v$

$$\begin{aligned}\nabla'\cdot\overline{\mathbf{Q}}(\mathbf{r},\mathbf{r}') &= i\omega\mu_0\overline{\mu}_r(\mathbf{r}')\cdot\mathbf{H}(\mathbf{r}')\cdot\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]-k_0^2\mathbf{E}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \\ &- i\omega\mu_0\mathbf{H}(\mathbf{r}')\cdot\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]+k_0^2\overline{\epsilon}_r(\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')-\mathbf{E}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\end{aligned} \tag{3}$$

where $(-i\omega\mu_0)(i\omega\epsilon_0)=k_0^2$. We will hereon refer to (2) as $\nabla'\cdot\overline{\mathbf{Q}}_0$ to distinguish it from (3).

Next, on integrating $\nabla'\cdot\overline{\mathbf{Q}}$ either over v_0 or over v separately, we avoid issues related to possible discontinuities of $\overline{\mathbf{Q}}$ and its derivatives across the boundary of v. Doing so and employing the divergence theorem and integrating over v we get

$$\int_s ds'\hat{n}'\cdot\overline{\mathbf{Q}}(\mathbf{r},\mathbf{r}')=\int_v dv'\overline{\mathbf{U}}(\mathbf{r},\mathbf{r}')-\begin{cases}\mathbf{E}(\mathbf{r}) & \mathbf{r}\in v\\ 0 & \mathbf{r}\in v_0\end{cases} \tag{4}$$

where,

$$\overline{\mathbf{U}}(\mathbf{r},\mathbf{r}')=i\omega\mu_0\left[\left(\overline{\mu}_r(\mathbf{r}')-\overline{\mathbf{I}}\right)\cdot\mathbf{H}(\mathbf{r}')\right]\cdot\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]+k_0^2\left[\left(\overline{\epsilon}_r(\mathbf{r}')-\overline{\mathbf{I}}\right)\cdot\mathbf{E}(\mathbf{r}')\right]\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'). \tag{5}$$

Alternatively, integrating (2) over v_0, with $s_0\to\infty$, and considering the two cases where $\mathbf{r}\in v_0$ or $\mathbf{r}\in v$, use of the divergence theorem gives

$$-\int_s ds'\hat{n}'\cdot\overline{\mathbf{Q}}_0(\mathbf{r},\mathbf{r}')=\mathbf{I}(\mathbf{r})-\begin{cases}\mathbf{E}(\mathbf{r}) & \mathbf{r}\in v_0\\ 0 & \mathbf{r}\in v\end{cases} \tag{6}$$

in which $-\hat{n}'$ is the outward normal to s and $\mathbf{I}(\mathbf{r})=i\omega\mu_0\int_{v_s}d\mathbf{r}'\mathbf{J}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\mathbf{E}^{inc}(\mathbf{r})$ is the excitation or the incident field generated by the source $\mathbf{J}(\mathbf{r})$. Adding (4) and (6) and invoking tangential field continuity across the surface of the dielectric volume s yields

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \mathbf{E}^{inc}(\mathbf{r})+\int_v dv'\left\{k_0^2\left[\overline{\epsilon}_r(\mathbf{r}')-\overline{\mathbf{I}}\right]\cdot\mathbf{E}(\mathbf{r}')\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')+i\omega\mu_0\left[\overline{\mu}_r(\mathbf{r}')-\overline{\mathbf{I}}\right]\cdot\mathbf{H}(\mathbf{r}')\cdot\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]\right\} \\ &= \mathbf{E}^{inc}(\mathbf{r})+\int_v dv'\left\{k_0^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\left[\overline{\epsilon}_r(\mathbf{r}')-\overline{\mathbf{I}}\right]\cdot\mathbf{E}(\mathbf{r}')-i\omega\mu_0\left[\nabla'\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]\cdot\left[\overline{\mu}_r(\mathbf{r}')-\overline{\mathbf{I}}\right]\cdot\mathbf{H}(\mathbf{r}')\right\}\end{aligned} \tag{7}$$

and is valid for both $\mathbf{r}\in v$ and $\mathbf{r}\in v_0$. It is seen that above expression does not contain any surface integrals or differentiation on material properties. Most importantly, no assumptions are made about the form of $\mathbf{E}$ and its derivatives across $s=\partial v$. The only requirement pertains to the enforcement of the natural tangential field continuity.

III. Numerical Implementation and Validation

Having derived the volume integral representation (7), we next proceed with its numerical implementation and validation. Specifically, in this section we present the Method of Moments (MoM) results when (7) is cast in the form $[Z_{ji}]\{x_i\} = \{b_j\}$.

Our first example is a composite cube of side length $0.2\lambda_0$, where λ_0 is the free-space wavelength of the incident plane wave. The incident plane wave is propagating in the negative z direction and the bistatic radar cross section of the cube is plotted in Fig. 1 (b). As seen, the agreement between the VIE and FE-BI solution is very good (for both polarizations) in calculating the bistatic radar cross section (RCS) for $\epsilon_r = 1.5$ and $\mu_r = 2.2$.

As a second example, we consider a thin spherical shell of outer radius of $0.2\lambda_0$, thickness $0.02\lambda_0$ and of the same material parameters ($\epsilon_r = 1.5, \mu_r = 2.2$) as in the previous example. For modelling, the mesh is constructed entirely as a thin layer of distorted (non-rectangular) curvilinear elements, and both VIE and FE-BI solution used the same mesh. The bistatic RCS is shown in Fig. 1 (c) showing full agreement between the two methods.

As our third example, we evaluate the RCS of a solid sphere having a radius $0.15\lambda_0$. The permittivity and permeability of the sphere are again $\epsilon_r = 1.5$ and $\mu_r = 2.2$, respectively. Using the same volumetric mesh as shown in Fig. 1 (d) for both methods, we evaluate the bistatic RCS. Again, excellent agreement is seen between the VIE and FE-BI solutions.

At the meeting, we will provide numerical details relating to the VIE and FE-BI methods. As usual, the FE-BI contains a partly sparse and partly full matrix, whereas the VIE leads to a fully populated system. The associated VIE is however of the 2^{nd} kind and this is an attractive feature for iterative solvers. As an example, for a system of 1082 (second example) VIE unknowns, the conjugate gradient squared (CGS) solver converged in 5 iterations for a relative error of 0.01 and the solution was completed in only 0.7 seconds using a 1 GHz PIII processor.

IV. Conclusions

This paper dealt with the derivation and validation of volume integral equations for modeling the scattering by volumes having non-trivial permittivity and permeability. The derived volume integral equation involved a single unknown (the electric or magnetic field). The goal was to demonstrate that no surface integral is needed at the boundaries where the permittivity and/or permeability are discontinuous. Although comparisons were given with the FE-BI method, there is no claim on the efficiency of either method. The quoted FE-BI results were given only for validation purposes.

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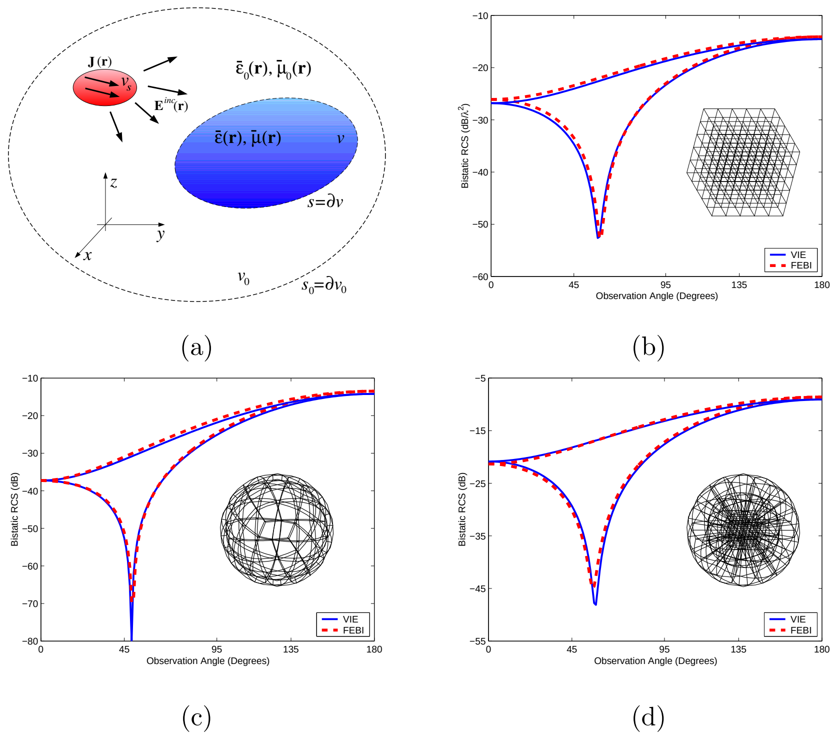

Fig. 1. (a) Geometrical setup for the volume integral equation derivations, (b) Bistatic radar cross section (RCS) of a homogeneous composite cube of side length $a = 0.2\lambda$, and $\epsilon_r = 1.5$, $\mu_r = 2.2$ (incidence at zero degrees), (c) Bistatic RCS of a composite spherical shell. The outer shell radius is $r_o = 0.2\lambda$, its thickness is $d = 0.02\lambda$, and its relative constitutive parameters are ($\epsilon_r = 1.5, \mu_r = 2.2$), (d) Bistatic RCS of a homogeneous composite sphere of radius $r = 0.15\lambda$. The sphere has the relative constitutive parameters ($\epsilon_r = 1.5, \mu_r = 2.2$)

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