

Fast Solution of Electromagnetic Boundary Value Problems by the Characteristic Basis Functions/FEM Approach

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1. INTRODUCTION

The Finite Element Method (FEM) is a powerful approach to the numerical solution of electromagnetic scattering and/or radiation problems [1]. The computational domain is decomposed into a number of elements of simple shape over which the unknown field quantities are represented in terms of pre-defined basis functions. The matrix equation, which results from the weak variational formulation, turns out to be sparse, which enables the utilization of effective direct or iterative matrix solution algorithms [2].

It is well-known that the element size in FEM formulations depends on the wavelength λ , and a generally accepted 'resolution rule' is to construct a mesh containing approximately ten nodes per wavelength for an accuracy of 1 % in the solution. As a result, for relatively high frequency applications, the number of degrees of freedom may become prohibitively large, and the numerical calculation may be unmanageable even with very powerful computers.

Iterative solution algorithms are well-suited for the solution of sparse FEM matrix equations, but in most cases the algorithms converge very slowly [3]. Preconditioners can be devised to circumvent this difficulty, but this is not an easy task in general. Despite these difficulties encountered in iterative solution techniques, direct approaches (designed for sparse matrices) are also not applicable in problems with a large number of unknowns, due to the limitations in computer storage and speed. Domain decomposition methods [4] have been introduced to handle large matrix equations resulting from the discretizations used in the finite methods. The basic idea in domain decomposition approach is to decompose the computational domain into a number of smaller subdomains. Then, the subdomains are treated as if they are 'isolated', and the subdomain couplings are further added via a direct or iterative algorithm. An attractive aspect of these methods is their implementation on parallel machines.

The method described in this paper resembles the domain decomposition method, but it exhibits some distinctive differences in the implementation. The computational domain (composed of the FEM mesh) is subdivided into a number of subdomains and a hierarchical class of characteristic basis functions (CBFs) are evaluated. The computation of the CBFs is based on the physics of the problem, since they are computed via the couplings of the subdomains. The hierarchical structure is based on the order of the couplings, as will be seen in the derivation of the algorithm.

2. CONSTRUCTION OF THE CBFs

In order to illustrate the algorithm, we start with a simple model problem, which is the construction of the two-dimensional Green's function where the partial differential equation is the well-known Helmholtz equation, as given below:

$$\nabla^2 u + k^2 u = -\delta(\vec{r}) \quad (1)$$

and the solution is given by $u(\vec{r}) = (j/4)H_0^2(k|\vec{r}|)$. Let Ω be the computational domain shown in Fig. 1. By using the symmetry in the problem only one-fourth of the real computational domain is used. In the free-space region Ω_{FS} , u satisfies the equation given above, and in the region occupied by the PML (Ω_{PML}), the partial differential equation is given by:

$$\nabla \cdot ([\Lambda] \nabla u + k^2 \gamma u) = 0 \quad (2)$$

where the matrix $[\Lambda]$ and the parameter γ are related to the properties of the PML medium.

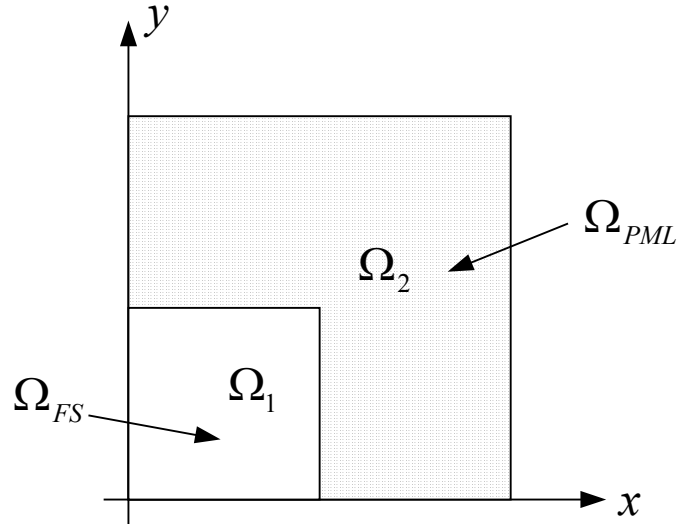


Fig. 1 The computational domain with Two Subdomains

For the sake of simplicity, two subdomains are defined in the computational domain as $\Omega_1 = \Omega_{FS}$ and $\Omega_2 = \Omega_{PML}$ as shown in Fig. 1. The finite element matrix becomes a matrix with 4 submatrices, and the discretized system can be written as:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

Since the point source is located at the origin, the right hand side vector is nonzero only in the domain Ω_1 .

Now, define the vectors v_i^j as follows (the superscript j denotes the step number and the subscript i denotes the subdomain)

- $v_1^1 = \begin{bmatrix} A_{11}^{-1} b_1 \\ 0 \end{bmatrix}$, $v_2^1 = \begin{bmatrix} 0 \\ A_{22}^{-1} 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (These are the primary basis vectors)

- $v_1^2 = \begin{bmatrix} A_{11}^{-1} A_{12} A_{22}^{-1} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_2^2 = \begin{bmatrix} 0 \\ A_{22}^{-1} A_{21} A_{11}^{-1} b_1 \end{bmatrix}$ (These are the secondary basis vectors)
- $v_1^3 = \begin{bmatrix} A_{11}^{-1} A_{12} A_{22}^{-1} A_{21} A_{11}^{-1} b_1 \\ 0 \end{bmatrix}$, $v_2^3 = \begin{bmatrix} 0 \\ A_{22}^{-1} A_{21} A_{11}^{-1} A_{12} A_{22}^{-1} 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (These are the third-order basis vectors)

It is evident that higher order basis vectors can be evaluated by using this algorithm. It should be noted that the zero vectors are discarded, which implies that only a single basis vector is added at each step. Therefore, we define the vectors w_j , $j = 1, 2, \dots, N_s$ as follows (N_s denotes the number of steps):

$$\begin{aligned} w_1 &= v_1^1 \\ w_2 &= v_2^2 \\ w_3 &= v_1^3 \\ &\vdots \\ &\text{etc.} \end{aligned}$$

Then, we express the solution x as:

$$x = \sum_{j=1}^{N_s} \alpha_j w_j$$

To evaluate the unknown coefficients we form the matrix equation:

$$\sum_{j=1}^{N_s} \alpha_j \langle Aw_j, w_i \rangle = \langle b, w_i \rangle, \quad i = 1, 2, \dots, N_s \quad (3)$$

3. NUMERICAL RESULTS

The computational domain Ω is the square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$ and the wavenumber is chosen as $k = \pi$. The free space region Ω_{FS} is the square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and the remaining part of the domain is the PML region Ω_{PML} .

We define the solution of the FEM equation (without partitioning) as x_{FEM} and the solution obtained by partitioning as x_{DD} (DD stands for domain decomposition). We define the percentage error as:

$$e(\%) = \frac{\|x_{FEM} - x_{DD}\|}{\|x_{FEM}\|} \times 100$$

We consider domains with different mesh-sizes, as follows:

1. The FEM mesh contains 256 nodes (mesh-size $h = 2./15$)
2. The FEM mesh contains 961 nodes (mesh-size $h = 2./30$)
3. The FEM mesh contains 2116 nodes (mesh-size $h = 2./45$)
4. The FEM mesh contains 3721 nodes (mesh-size $h = 2./60$)

Table 1. Performance of the algorithm with different mesh sizes.

N_s	$e(\%)$ 256 nodes	$e(\%)$ 961 nodes	$e(\%)$ 2116 nodes	$e(\%)$ 3721 nodes
2	84.53	87.94	89.45	89.64
3	49.99	54.20	55.33	56.09
4	43.02	50.00	52.18	53.70
5	11.32	21.49	24.83	29.08
6	6.54	16.57	20.49	25.12
7	1.93	7.08	10.82	15.41
8	0.71	4.26	7.39	11.47
9	0.22	1.96	3.69	6.35
10	0.04	1.04	2.34	4.39
11		0.57	1.39	2.65
12		0.25	0.77	1.71
13		0.13	0.46	1.02
14		0.05	0.23	0.63
15			0.15	0.42
16			0.06	0.24
17				0.15
18				0.12
19				0.10
20				0.08

We observe that the number of basis vectors increases only slightly when the mesh-size is reduced. We have also found that the same is true when the size of the problem is increased while leaving the mesh size unchanged, i.e., the number of CBFs grow very slowly with the increase in the size, enabling us to solve large problems directly without resorting to iteration.

REFERENCES

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