A note on DeRham diagram, tree-cotree splitting, and Schwarz preconditioners

Jin-Fa Lee
ESL, Electrical Engineering Dept.
1320 Kinnear Rd.
The Ohio State University
Lee.1863@osu.edu

In this talk, I shall detail three recent concepts/algorithms that are of paramount importance in solving PDEs efficiently. Two major issues are at the heart of developing robust and efficient FEM solvers for Maxwell equations: The well-posedness of the formulation throughout the frequency range; and, The robust and efficient preconditioner render the system matrix equations easy to solve. To address these two questions directly, I shall discuss in detail three topics that are closely related.

1. Basis functions. An overview of basis functions, employed in vector finite element methods (FEMs) as well as volume integral equations (VIEs), will be described in simple differential forms through the de-Rham diagram. It will be emphasized that in both the FEMs and integral equation formulations, the traditional approaches usually suffer the so-called low frequency instability. The low frequency instability problem results in two important consequences: **a**. The numerical solutions do not improve as the mesh get refined; and, **b**. At very low frequency applications, the numerical solutions are either very wrong or the iterative matrix solution technique fails to converge. Remedies such as tree-cotree splitting (in FEM) and loop-star basis functions (in MOM) will be discussed.
2. Tree-cotree splitting. A very efficient preconditioner based on the multiplicative Schwarz method employed in a p-type environment has been developed in the author's group in recent years. The idea is to derive the higher order basis functions in a hierarchical way, and treating each polynomial function space as a "domain". However, the p-version of the Schwarz methods strongly depend on the basis functions possess the loosely coupled nature. Subsequently, it is of paramount importance in splitting the curl-conforming vector basis functions through an in-exact Helmholtz decomposition process. Namely, the basis functions need to be written explicitly as pure gradient functions and in-exact "curl-like" functions. As it turns out, the splitting of the lowest-order curl-conforming basis functions, edge elements, is also essential. Without the tree-cotree splitting at the edge elements level, the solution process will breakdown due to a low-frequency instability problem suffered by conventional field formulation (R. Dyczij-Edlinger, G. Peng, and J.F. Lee, CMAME, 169, Feb 1999). Details of how to construct the tree-cotree splitting from a basis graph theory will be presented.
3. Schwarz preconditoners. Two types of Schwarz preconditioners are common in recent years in solving PDEs. The additive and the multiplicative Schwarz methods. The multiplicative Schwarz method not only takes fewer iterations to converge (usually ½) than the additive version but also much more robust (it succeeds when additive version fails). A simple error bounds for both the geometric multigrid and the p-type Schwarz method will be presented in the talk.