A FIT Matrix Formulation for the Application of Global Boundary Conditions in Frequency Domain

A. Skarlatos[1*], R. Schuhmann, and T. Weiland
Technische Universität Darmstadt
Computational Electromagnetics Laboratory (TEMF)
Schloßgartenstr. 8, 64289 Darmstadt, Germany
skarlatos@temf.tu-darmstadt.de, schuhmann@temf.tu-darmstadt.de, weiland@temf.tu-darmstadt.de

Abstract

A global boundary condition based on an integral representation of the boundary electric field is derived for the Finite Integration Technique in Frequency Domain. The boundary condition is expressed in form of a discrete boundary operator which conforms with the FIT matrix-based approach. For the solution of the overall system of equations an iterative technique is used.

I. INTRODUCTION

Global Boundary Conditions in Finite Differences schemes based on integral representations are usually not preferred because they lead to dense matrices. This is a major disadvantage especially when the problem has to be solved implicitly, e.g. in a frequency domain (FD) formulation. The hybridization of Finite Differences with integral or asymptotic methods, however, leads inevitably to such boundary conditions. In this paper a global boundary condition for the Finite Integration Technique - FIT (an alternative formulation of the finite differences method [1]) in FD is derived. The boundary condition is expressed in form of a matrix equation providing compact formulation (conformal with the FIT matrix-approach) and hiding the details of its evaluation. Since the geometry of the exterior domain and the grid are given, the boundary matrices can be evaluated independently from the interior problem itself.

Instead of trying to solve the overall system formed by the curl-curl equation and the boundary condition (which is a difficult task because of the density of the boundary matrix) an iterative algorithm is used, where each equation is treated independently. This process is repeated until a satisfactory accuracy is obtained.

II. THE GLOBAL BOUNDARY CONDITION FOR THE FIT-FD

In contrast to the FEM, where the coupling with the exterior domain is carried out by the surface integral of the weak formulation, in FIT the boundary condition has to be explicitly posed. According to the equivalence principle, the field on the boundary can be expressed as an integral representation of the tangential fields on any arbitrary surface A enclosed by the boundary. The tangential electric field on the boundary can then be written

$$\begin{aligned}\vec{n}\times\vec{E}\left(\vec{r}\right) &= \vec{n}\times\oint_A\left\{\overline{\overline{G}}{}_J^E\left(\vec{r},\vec{r}'\right)\cdot\left[\vec{n}\times\vec{H}\left(\vec{r}'\right)\right]\right.\\ &- \left.\overline{\overline{G}}{}_M^E\left(\vec{r},\vec{r}'\right)\cdot\left[\vec{n}\times\vec{E}\left(\vec{r}'\right)\right]\right\}dA'+\vec{n}\times\vec{E}^{inc}(\vec{r}), \qquad (1)\end{aligned}$$

where $\overline{\overline{G}}{}_J^E\left(\vec{r},\vec{r}'\right)$, $\overline{\overline{G}}{}_M^E\left(\vec{r},\vec{r}'\right)$ are the Green functions for the electric and magnetic current sources, and $\vec{E}^{inc}(\vec{r})$ is the incident field (expressing the contribution of radiation sources located in the exterior domain). Supposing the Green functions are known (or can be computed numerically), (1) forms the boundary condition we ask for. As integration (Huygens) surface A we choose the box defined by the primary grid planes one grid cell behind the boundary (see Fig. 2). Substituting the modified Green Functions of [2] in (1), FIT can be coupled with the Uniform Theory of Diffraction (UTD) to model the influence of large objects lying outside the calculation domain.

1. Supported by the State Scholarship's Foundation of Greece

III. The Modified Curl-Curl Equation

The discretization of the Maxwell Equations according to the FIT scheme leads to the Maxwell Grid Equations:

$$\begin{aligned} \mathbf{C}\widehat{\mathbf{e}} &= -j\omega\widehat{\widehat{\mathbf{b}}} &(2)\\ \widetilde{\mathbf{C}}\widehat{\mathbf{h}} &= j\omega\widehat{\widehat{\mathbf{d}}} + \widehat{\widehat{\mathbf{j}}}_s, &(3) \end{aligned}$$

where $\mathbf{C}$, $\widetilde{\mathbf{C}}$ are topological matrices, forming an algebraic analogous of the continuous curl operator in the vector spaces defined by the primary and the dual grid respectively [1]. The electric and magnetic fluxes $\widehat{\widehat{\mathbf{d}}}$ and $\widehat{\widehat{\mathbf{b}}}$ through the grid facets are related to the corresponding voltages along the grid edges $\widehat{\mathbf{e}}$ and $\widehat{\mathbf{h}}$ via the material matrices (linear case):

$$\begin{aligned} \widehat{\widehat{\mathbf{b}}} &= \mathbf{M}_\mu\widehat{\mathbf{h}} &(4)\\ \widehat{\widehat{\mathbf{d}}} &= \mathbf{M}_\varepsilon\widehat{\mathbf{e}}. &(5) \end{aligned}$$

Combining (2),(3) and (4),(5), we obtain the discrete form of the curl-curl equation:

$$(\mathbf{M}_\varepsilon^{-1}\widetilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{C} - \omega^2\mathbf{I})\widehat{\mathbf{e}} = -j\omega\mathbf{M}_\varepsilon^{-1}\widehat{\widehat{\mathbf{j}}}_s. \tag{6}$$

The electric and the magnetic boundary conditions can be embedded in the material matrices by setting the corresponding entries in $\mathbf{M}_\varepsilon^{-1}$ to zero. In order to apply an arbitrary boundary condition, however, (6) has to be modified as follows

$$(\mathbf{M}_\varepsilon^{-1}\widetilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{C} - \omega^2\mathbf{I})\widehat{\mathbf{e}} = -j\omega\mathbf{M}_\varepsilon^{-1}\widehat{\widehat{\mathbf{j}}}_s - \omega^2\widehat{\mathbf{e}}_b, \tag{7}$$

where $\widehat{\mathbf{e}}_b$ is the vector of electric voltages tangential to the boundary, and the material matrices correspond to an electric boundary condition. The tangential electric voltage is supposed to be given and it is thus placed on the right hand side of the above relation, acting as a magnetic current source. Recalling that an electric boundary condition was assumed during the construction of the material matrices, (7) reduces at the boundary edges (index i) to

$$\omega^2\widehat{e}_i = \omega^2\widehat{e}_{b_i}. \tag{8}$$

Here the equivalence principle is applied in the inverse direction, namely the tangential fields on the boundary describe the exterior domain. Actually both the electric and the magnetic field (or equivalently magnetic and electric currents) on the boundary must be given in order the problem to be uniquely defined. By filling however the external region with a PEC material (electric boundary condition in the construction of the material matrices) the electric currents (magnetic field) on the boundary are short-circuited and they can be thus omitted [3].

In the next step we must calculate the boundary electric field $\widehat{\mathbf{e}}_b$. This can be done applying (1). In the discrete space of the FIT grids the integration of (1) becomes a summation and the continuous relation takes the following matrix form:

$$\widehat{\mathbf{e}}_b = \mathbf{U}_\mathbf{e}\widehat{\mathbf{e}} + \mathbf{U}_\mathbf{h}\widehat{\mathbf{h}} + \widehat{\mathbf{e}}_{inc}. \tag{9}$$

The $\mathbf{U}_\mathbf{e}$ and $\mathbf{U}_\mathbf{e}$ matrices can be considered as discrete operators that map the electric and magnetic voltages of the FIT domain to the electric voltages on the boundary. Every non-zero line corresponds to a boundary value and contains the summation coefficients of the voltages on the Huygens surface arising from the discretization of the integral in (1). Substituting the magnetic field from the grid Faraday equation (2) we have:

$$\widehat{\mathbf{e}}_b = \left(\mathbf{U}_\mathbf{e} - \frac{1}{j\omega}\mathbf{U}_\mathbf{h}\mathbf{M}_\mu^{-1}\mathbf{C}\right)\widehat{\mathbf{e}} + \widehat{\mathbf{e}}_{inc}. \tag{10}$$

It should be noticed that in FIT the two fields are allocated on staggered grids and one of them must be interpolated. In our case the integration surface was chosen to coincide with the primary grid planes and thus the magnetic field has to be interpolated on this plane. The corresponding interpolation scheme is also included in $\mathbf{U}_\mathbf{h}$ in (9)

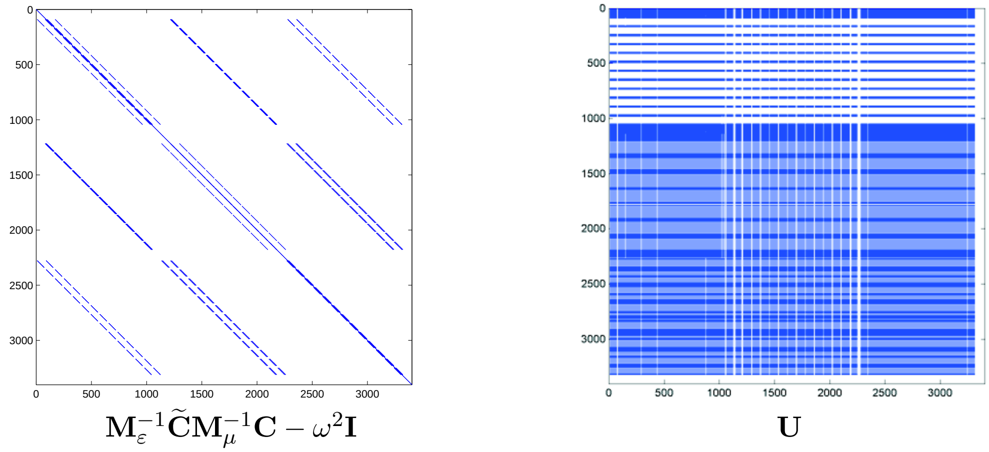

Fig. 1. The structure of the matrices: The curl-curl system matrix and the boundary matrix for a $9 \times 9 \times 14$ grid. The higher the ratio of the boundary edges to the total edges is, the denser the matrix becomes. In the specific example this ratio is relatively high.

IV. The Solution of the Equation System

As showed above, the problem is completely defined by the following set of equations:

$$
\begin{aligned}
(\mathbf{M}_\varepsilon^{-1}\widetilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{C}-\omega^2\mathbf{I})\widehat{\mathbf{e}} &= -j\omega\mathbf{M}_\varepsilon^{-1}\widehat{\widehat{\mathbf{j}}}_s-\omega^2\widehat{\mathbf{e}}_b && (11)\\
\widehat{\mathbf{e}}_b &= \mathbf{U}\widehat{\mathbf{e}}+\widehat{\mathbf{e}}_{inc}. && (12)
\end{aligned}
$$

The boundary matrix $\mathbf{U}=\mathbf{U}_\mathbf{e}-\frac{1}{j\omega}\mathbf{U}_\mathbf{h}\mathbf{M}_\mu^{-1}\mathbf{C}$, however, has many non-vanishing entries which makes the overall system difficult to be solved (cf. Fig. 1). To overcome this difficulty and to avoid the inversion of $\mathbf{U}$ the set of the two equations is solved using an iterative technique. Initially the curl-curl equation is solved assuming a simple electrical boundary condition. The solution is substituted into (12) to provide a first approximation for the field on the boundary. This result is used then as a more precise boundary condition in the curl-curl equation which is solved again to obtain a better approximation of the real solution. This process is repeated until the overall residual falls under a given limit. The speed of the above technique depends on the time needed for the solution of the curl-curl equation with an electric boundary condition. This problem however is well conditioned and thus converges fast. It is solved using an iterative technique (BiCG seems to be a good choice in this case) where the solution of the previous iteration can be used as an initial guess for the current iteration, providing a slight improvement in its convergence.

V. Validation

The above described boundary condition is applied to the calculation of the radiation of a dipole of $15\,cm$ length in the presence of a 90° corner reflector (cf. Fig. 3). The dimensions of the FIT domain enclosing the dipole are $10\,cm \times 10\,cm \times 20\,cm$. The impact of the reflector is incorporated into the Green Functions used by the boundary condition as described in [2]. Fig. 4 shows the results obtained with the presented boundary condition, compared to the results given by FIT using a PML for the truncation of the computational domain (the reflectors are modelled via an electrical boundary condition).

VI. Conclusions

In this paper a matrix formulation for the application of global boundary conditions to FIT-FD problems was derived. It enables the hybridization of FIT with the UTD and other methods by using modified Green Functions. For the solution of the system an iterative algorithm is used. Its properties and the effect of the application of different solvers will be a subject of further research.

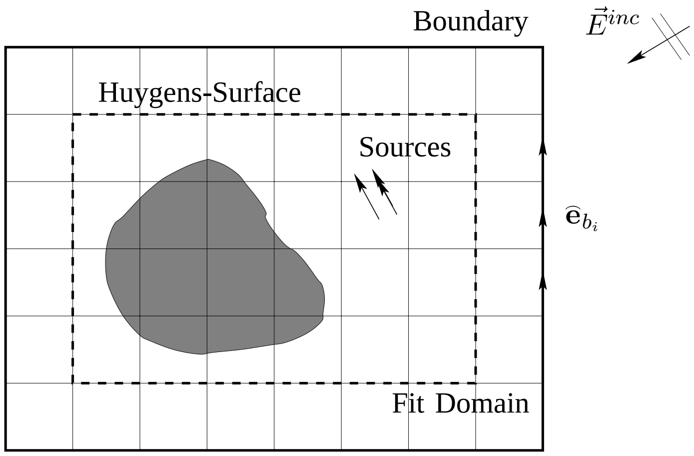

Fig. 2. Integral representation of the Boundary fields.

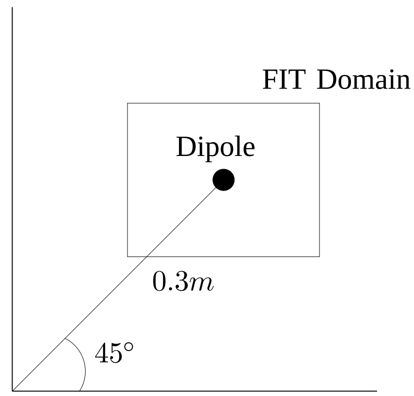

Fig. 3. Validation example: Dipole in the presence of a corner reflector.

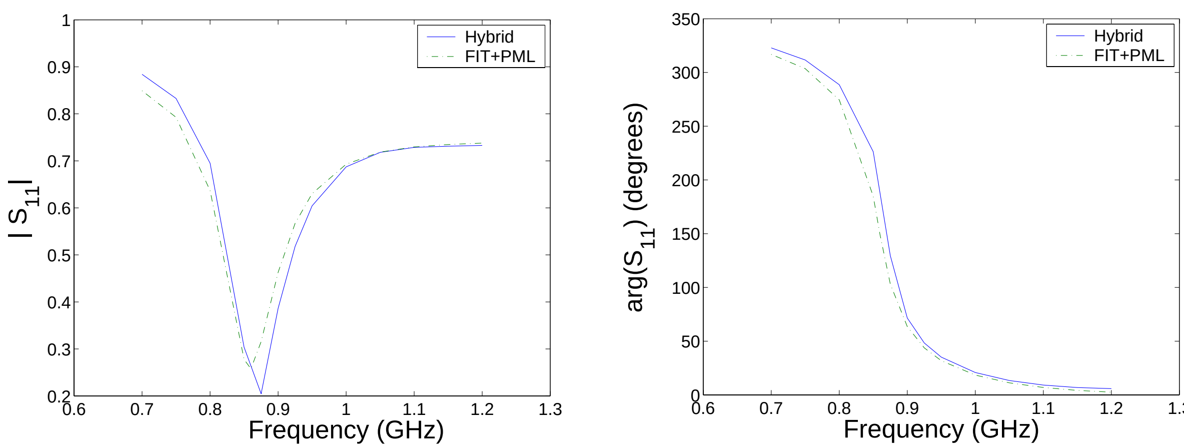

Fig. 4. Amplitude and phase of the S_{11} parameter of the dipole around the first resonance.

References

[1] T. Weiland, “Time domain electromagnetic field computation with finite difference methods,” *Int. J. Num. Mod.* vol. 9, pp. 259-319, 1996.

[2] M. Alaydrus, V. Hansen, and T. F. Eibert, “Hybrid^2: Combining the three-dimensional hybrid finite elements - boundary integral technique for planar multilayered media with the uniform geometrical theory of diffraction,” *IEEE Trans. Antennas Propag.*, vol. 50, pp. 67-74, Jan. 2002

[3] C. A. Balanis, “ Advanced engineering electromagnetics.” New York: John Wiley & Sons, 1989.