Stable Waveguide Ports for the Time Domain Finite Element Method

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1 Introduction

In many microwave applications, the computation of scattering parameters requires reliable boundary conditions for the truncation of waveguide structures. The finite element method [1] (FEM), often formulated in the frequency domain, is a popular choice for this class of problems. Recently, Alimenti et al. [2] revised a formulation of modal absorbing and matched modal source boundary conditions for the finite-difference time-domain (FDTD) scheme. However, the basic FDTD scheme is constructed from Cartesian grids and, consequently, waveguides with cross sections which do not conform to the underlying structured grid will suffer from the staircase approximation.

In this article, we introduce a new waveguide port algorithm for transient FEM computations. The waveguide port algorithm is suitable for homogeneous waveguides of arbitrary cross section as well as similar problems in e.g. acoustics. Assuming that a sufficient number of waveguide modes is used, the suggested algorithm does not give any reflection from the waveguide port. This result holds for all frequencies supported by the FEM discretization. We demonstrate the new waveguide port algorithm by using it in conjunction with the stable FEM-FDTD hybrid [3, 4] for the computation of the scattering parameter of a patch antenna fed by a coaxial cable. The coaxial cable and a small volume around the feed are discretized by the FEM and the remaining large volume (including the major part of the substrate) is discretized by the FDTD scheme.

2 Formulation

We discretize the cross section of the waveguide by triangles and/or quadrilaterals. These elements are extruded a distance h perpendicular to the plane of the cross section and, thus, they form the bases to straight prisms and/or hexahedrons, respectively. The volume elements are copied repeatedly along the waveguide axis, which gives a periodic discretization of an infinitely long waveguide. For this setting, we align the z-axis of a Cartesian coordinate system with the cylindrical axis of the waveguide. The unit layer of finite elements for a coaxial cable is shown in Fig. 1(a).

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For this setup, we express the electric field in Maxwell's equations $\nabla \times \mu_r^{-1} \nabla \times \vec{E} = (\omega/c_0)^2 \epsilon_r \vec{E}$ in terms of the edge elements [1], i.e. $\vec{E}(\vec{r}) = \sum_{j=1}^{\infty} E_j \vec{N}_j(\vec{r})$. Here, μ_r and ϵ_r can be functions of x and y. Next, we impose a Floquet representation with an $\exp(-jk_z z)$ dependence to connect the cells separated by integer multiples of h in the z-direction. We employ Galerkin's method and test with the edge elements $\vec{N}_i(\vec{r})$ associated with the unit layer of degrees of freedom shown in Fig. 1(b). This procedure results in a generalized eigenvalue problem $\mathbf{S}(k_z)\mathbf{E} = (\omega/c_0)^2 \mathbf{M}(k_z)\mathbf{E}$, which can be solved for the dispersion relations $\omega = \omega(k_z)$ and the corresponding eigenmodes $\mathbf{E} = \mathbf{E}(k_z)$. We number the waveguide modes with the indices $m = 1, \ldots, M$ where M is the number of test edges. For waveguides with homogeneous cross section, the modes $\mathbf{E}$ are constant with respect to k_z and the stiffness matrix $\mathbf{S}$ and mass matrix $\mathbf{M}$ are Hermitian. In the following, we limit the discussion to the case when the permittivity and permeability are constants.

For a waveguide discretized by finite elements, we can construct the transmission line equation by so-called macro elements $\vec{M}_j(\vec{r}; m) = \sum_i E_i^{(m)} \vec{N}_i(\vec{r})$, where $E_i^{(m)}$ are the coefficients for the m-th mode computed from the eigenvalue problem above. The basis function of the macro element $\vec{M}_j(\vec{r}; m)$ is non-zero for $(j-1)\Delta z < z < (j+1)\Delta z$, where j is an integer.

We express the electric field in terms of standard edge elements $\vec{N}_j(\vec{r})$ for the part of the computational domain where the solution has a three-dimensional dependence. For the part of a waveguide which is treated as a one dimensional problem, we expand the electric field for the m-th mode in terms of the macro elements $\vec{M}_j(\vec{r}; m)$. Thus, the total field is $\vec{E}(\vec{r}, t) = \sum_j E_j(t) \vec{N}_j(\vec{r}) + \sum_{jm} E_j^{(m)}(t) \vec{M}_j(\vec{r}; m)$.

Following Galerkin's method, we test Maxwell's equations by all $\vec{N}_j(\vec{r})$ and $\vec{M}_j(\vec{r}; m)$. This gives a system of coupled ordinary differential equations with respect to time and we use the time-stepping scheme [4]:

$$\sum_{k=1}^{K} \left(\mathbf{S}_k \left[\theta_k \mathbf{E}^{n+1} - (2\theta_k - 1)\mathbf{E}^n + \theta_k \mathbf{E}^{n-1} \right] + \mathbf{M}_k \left[\frac{\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}}{(c_0 \Delta t)^2} \right] \right) = 0, \quad (1)$$

where we assign the implicitness parameter θ_k to each (standard or macro) element k where $k = 1, \ldots, K$ and K is the total number of elements. Here, $\mathbf{S}_k$ denotes the contribution to $\mathbf{S}$ from element k, i.e. $\mathbf{S} = \sum_{k=1}^{K} \mathbf{S}_k$. The same partition is applied to the mass matrix $\mathbf{M}$. Unconditionally stable time-stepping is guaranteed for $\theta_k \geq 1/4$ which is used for the tetrahedrons and pyramids. In the part of the computational domain where we solve for the three-dimensional dependence of the fields, we lump the stiffness- and mass-matrices for the cubes and choose $\theta_k = 0$, which allows explicit time-stepping. Similarly, the prisms and/or hexahedrons where the fields are expressed as a sum of waveguide modes are lumped along the cylinder axis of the waveguide and, again, $\theta_k = 0$ allows explicit time-stepping. We terminate the one dimensional transmission line for the m-th mode by the scheme proposed by Alimenti et al. [2]. Stability of the entire scheme follows from the proof of stability constructed by Rylander and Bondeson [4].

3 Numerical examples

We test the stable waveguide port algorithm for a patch antenna. The patch is placed on a dielectric substrate ($\epsilon_r = 2.5$) which is backed by a perfect electric conductor (PEC) ground plane as shown in Fig. 2(a). The coaxial cable which feeds the antenna is discretized as shown in Fig. 1 where the outer radius is 3 mm and the inner radius is 0.48 mm. The dielectric of the coaxial cable is characterized by $\epsilon_r = 1.86$. The length of the coaxial cable is 33 mm and we express the field in terms of the fundamental TEM-mode for the part of the waveguide where amplitude of the higher (evanescent) modes is negligeble.

The computed reflection coefficient S_{11} is shown in Fig. 2(b) by diamonds, circles, and squares for three FDTD cell sizes $h_{\text{FDTD}} = 3.50$ mm, 2.33 mm, and 1.75 mm, respectively. (The unstructured finite element grid is refined similarly.) We model the fundamental mode of operation for the antenna by a lumped series resonance circuit with the impedance $Z_a = R + j\omega L + (j\omega C)^{-1}$. The antenna impedance Z_a terminates the coaxial cable transmission line with the characteristic impedance $Z_0 = 50\ \Omega$ and the length d. We evaluate the reflection coefficient at the port from $S_{11} = (Z_a|_{z=d} - Z_0)/(Z_a|_{z=d} + Z_0)$, where $Z_a|_{z=d} = Z_0(Z_a + jZ_0 \tan(\beta d))/(Z_0 + jZ_a \tan(\beta d))$ and β is the waveguide wavenumber. We use the three computed (complex) S_{11} around the resonance to determine R, L, C, and d. The models are shown in Fig. 2(b) by dash-dotted, dashed, and solid lines for the three different FDTD cell sizes $h_{\text{FDTD}} = 3.50$ mm, 2.33 mm, and 1.75 mm, respectively. For comparison, we include results computed by the transmission line method implemented in QuickWave3d and it is shown by the solid line marked with triangles in Fig. 2(b). It should be mentioned that these results are for finite cell sizes and some differences are expected.

The lumped circuit values are shown in Tab. 1 for the different h_{FDTD}. The Q-values are computed from $\sqrt{\gamma^2 + \omega_0^2}/2\gamma$, where $\omega_0 = \sqrt{1/LC - (R/2L)^2}$ is the resonant frequency and $\gamma = -R/2L$ is the damping of the lumped resonance circuit.

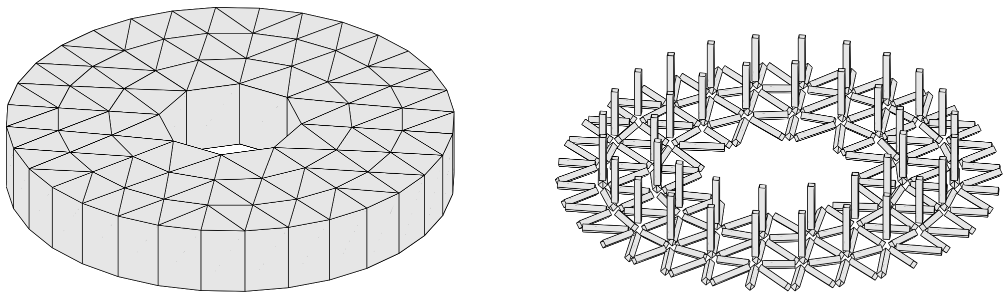
References

[1] J. M. Jin, *The Finite Element Method in Electromagnetics (2nd Edition)*. New York, NY: John Wiley & Sons, 2002.

[2] F. Alimenti, P. Mezzanotte, L. Roselli, and R. Sorrentino, "A revised formulation of model absorbing and matched modal source boundary conditions for the efficient FDTD analysis of waveguide structures," *IEEE Trans. Microwave Theory Tech.*, vol. 48, pp. 50–59, January 2000.

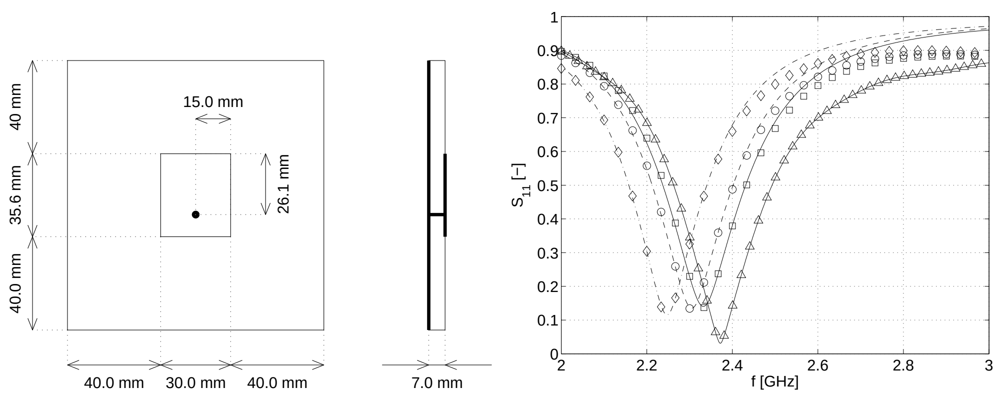
[3] T. Rylander and A. Bondeson, "Stable FEM-FDTD hybrid method for Maxwell's equations," *Comput. Phys. Comm.*, vol. 125, pp. 75–82, March 2000.

[4] T. Rylander and A. Bondeson, "Stability of explicit-implicit hybrid timestepping schemes for Maxwell's equations," *J. Comput. Phys.*, vol. 179, pp. 426–438, 2002.



(a) (b)

Figure 1: (a) Finite elements and (b) edges associated with a unit layer for a coaxial cable.



(a) (b)

Figure 2: (a) The geometry of the antenna. (b) Reflection coefficient S_{11}: QuickWave3d - curve with triangles; and stable FEM-FDTD hybrid for different cell sizes - the remaining curves and glyphs.

h_{FDTD} [mm]	R [Ω]	L [nH]	C [fF]	d [mm]	f_0 [GHz]	Q [-]
3.50	63.07	55.40	90.54	21.87	2.245	12.40
2.33	65.65	54.08	88.14	22.93	2.303	11.93
1.75	66.34	52.97	87.91	23.42	2.330	11.70

Table 1: Lumped circuit values together with the resonance frequency and the Q-value of the antenna.

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