Speed up the hybrid FEM+IE formulation using a low-rank matrix approximation

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I. Introduction

This paper addresses a methodology for hybridizing the Finite Element Method with the Integral Equation method FEM/IE for unbounded scattering or radiation problems in electromagnetics as well as speed up the IE portion of the computation using a single-level low-rank matrix approaximation. The motivation for this research has been the challenging problem of analyzing electrically large EM problems that contain heterogeneous bodies such as scattering by aircrafts and ships and radiation by large antenna arrays.

During the last decades much effort has been placed into the development of hybrid techniques for the analysis of EM problems. In particular the FEM/IE hybrid has attracted much of the interest. The currently existing hybrid FEM/IE methodologies can be roughly categorized into three groups:

1. Pure FEM/IE formulation based on a single outer boundary. In this method the FEM domain is truncated through a surface integral equation that describes the unbounded problem exterior to the FEM domain. In this category belongs the early work of Yuan [1] and Jin and Volakis [2][3]. The FEM/IE has been quite popular in the EM community due to its high accuracy and reliability. The major drawback is the resulting dense block in the final matrix equation that degrades the efficiency of the sparse iterative solvers. Moreover, the method is susceptible to the interior resonance problem that can be quite severe in electrically large structures.
2. Iterative Robin boundary condition through an IE formulation. In this method two truncation boundaries are used, one outer for the FEM domain, and one inner for the IE exterior domain. The outermost FEM boundary is truncated with the use of Robin boundary condition. The Robin boundary condition is updated by the IE evaluation for every node of the outermost boundary at every iteration step. Representative papers of this methodology could be found in references [4][5][6]. This method preserves the sparsity and symmetry of the finite element matrix; therefore efficient sparse matrix storage and solvers can be used. In addition, the method is free of interior resonances. On the other hand, the matrix equation needs to be solved for each iteration step since it varies from iteration to iteration. Another drawback is that in certain cases, the process may not converge.
3. Robin boundary condition with residual correction on two outer boundaries. Again, the Robin boundary condition is employed to truncate the FEM region to the exterior region. However, the IE evaluation is used to compute the residual that is needed to update the solution for the next iteration. A variant of this approach was used by Liu and Jin in [7]. The method seems quite attractive since the matrix equation remains the same from iteration to iteration. Only the residual needs to be computed for every iteration from the IE. The major inconvenience of this approach is the need of two boundaries, one for the sources and one for the observations.

Two major issues still remain in the hybrid FEM/IE formulation. They are how to speed up the IE portion of the computation and the development of proper preconditioner for the resulting matrix equations. A brute force implementation of FEM/IE results in $O\left(M^2\right)$ complexity, where M is the number of unknowns on the boundary. For a regular 3 dimensional object, assuming n discretization along each direction, we have $M \propto n^2 \propto N^{2/3}$. Thus the computational cost for the IE portion, just to assemble the matrix will be $\propto O\left(N^{4/3}\right)$. The overall computational effort can be even more expensive if the preconditioner of the resulting system requires the factorization of the IE matrix. Therefore, in the literature, various authors have successfully incorporated the FMM algorithm into the hybrid FEM/IE formulation to significantly reduce the computational cost especially for electrically large electromagnetic problems. There is only one problem. The codes that are written can be only used for one application, or one kind of Green's function, that is free space Green's function. If the FEM/IE-FMM formulation needs to

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be extended to other applications, such as scatterings with an infinite ground plane or finite antenna arrays in multiplayer medium environments, the modifications of the IE-FMM can be non-trivial. It is for this reason, we adopt a novel single-level low-rank approximation algorithm to speed up the IE computation. The approach depends only on the assumption that off-diagonal blocks of the IE portion of the matrix are effectively rank-deficiency. Consequently, the same algorithm can be applicable to various applications with little or no modifications.

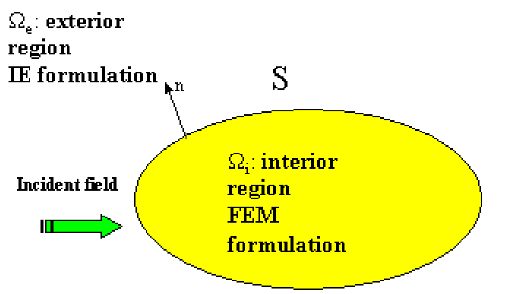


Figure 1: FEM/IE formulation. A single boundary S separates the entire problem domain into interior and the exterior regions.

II. Formulation

The boundary value problem (BVP) of the FEM/IE formulation can be stated as

$$\begin{aligned}
&\nabla\times\frac{1}{\mu_r}\nabla\times\vec{E}-k^2\varepsilon_r\vec{E}=0 \qquad \text{in } \Omega_i\\
&\left.\begin{aligned}&\nabla\times\nabla\times\vec{E}-k^2\vec{E}=-j\omega\mu_0\vec{J}^{inc}\\&\left(\nabla\times\nabla\times\vec{H}-k^2\vec{H}=\nabla\times\vec{J}^{inc}\right)\end{aligned}\right. \qquad \text{in } \Omega_e\\
&\lim_{r\to\infty} r\left(\nabla\times\vec{E}+jk\hat{r}\times\vec{E}\right)=0\\
&\hat{n}\times\vec{E}\big|_{S^-}=\hat{n}\times\vec{E}\big|_{S^+} \quad \text{and} \quad \hat{n}\times\nabla\times\vec{E}\big|_{S^-}=\hat{n}\times\nabla\times\vec{E}\big|_{S^+}
\end{aligned} \tag{1.1}$$

The interior portion of the formulation results in the following bilinear form

$$\int_{\Omega_i}(\nabla\times\vec{v})\frac{1}{\mu_r}\bullet\left(\nabla\times\vec{E}\right)-k^2\varepsilon_r\vec{v}\bullet\vec{E}dv' - jk\eta\int_S \vec{v}\bullet\vec{J}\left(\hat{n}\times\vec{E}\big|_S\right)ds' = 0 \tag{1.2}$$

Note that in equation (1.2), there are a few things worthy of mentioning: a. The trial vector function $\vec{E}$ needs to be curl-conforming as well as the test vector function $\vec{v}$; and, b. We have assumed that the surface current density, $\vec{J}\big|_S$, will be related to the tangential components of $\vec{E}$ on the surface S. The relation between $\vec{J}\big|_S$ and $\hat{n}\times\vec{E}\big|_S$ will be provided through integral equations and used as the boundary condition in (1.2).

A. EFIE Formulation

The electric field integral equation relates $\vec{J}\big|_S$ and $\hat{n}\times\vec{E}\big|_S$ by

$$\frac{1}{2}\hat{n}\times\vec{E}(\vec{r})=\hat{n}\times\vec{E}^{inc}(\vec{r})+\hat{n}\times\left(\oint_S\left(\hat{n}'\times\vec{E}(\vec{r}')\right)\times\nabla'g\,ds'+\oint_S\vec{J}(\vec{r}')g\,ds'-\frac{\nabla}{k^2}\oint_S\left(\nabla'\bullet\vec{J}\right)g\,ds'\right) \tag{1.3}$$

Equation (1.3) provides the necessary information to relate $\vec{J}\big|_S$ to $\hat{n}\times\vec{E}\big|_S$. Again, let us look at the smooth requirements: a. Since $\vec{E}$ is tangentially continuous, we have $\hat{n}\times\vec{E}$ be div-conforming (normal continuous); b. The surface current density $\vec{J}\big|_S$ involves divergence in (1.3) and should be expanded by div-conforming basis functions; and, c. Since we desire to write $\vec{J}\big|_S$ in terms of $\hat{n}\times\vec{E}\big|_S$, we shall test (1.3) by curl-conforming vector basis functions. After applying curl-conforming basis functions for testing, equation (1.3) results in a matrix equation of the form

$$P_E e_s + Q_E j_s = y_E \qquad (1.4)$$

B. MFIE Formulation

Similarly the relationship between $\vec{J}\big|_S$ and $\hat{n}\times\vec{E}\big|_S$ can be furnished by magnetic field integral equation (MFIE) and it is

$$\frac{1}{2}\vec{J}(\vec{r}) = \vec{J}^{inc}(\vec{r}) + \hat{n}\times\left(\oint_S \vec{J}(\vec{r}')\times\nabla' g ds' + k^2\oint_S \vec{M}(\vec{r}') g ds' + \oint_S \vec{M}(\vec{r}')\bullet\nabla' g ds'\right) \qquad (1.5)$$

It should be noted that if MFIE is the equation to be used, the surface current density $\vec{J}\big|_S$ does not need to be smooth at all. Matter of fact, just square-integrable will be sufficient to implement (1.5).

C. CFIE Formulation

As is well known that either EFIE or MFIE used alone will suffer interior resonances. This will particularly sever when the problem domain is electrically large. We adopt combine field integral equation (CFIE) in our formulation. However, this does have an implication: that is in the MFIE, the surface current density $\vec{J}\big|_S$ needs be div-conforming in order to be consistent with EFIE. Therefore, we shall test (1.5) also by div-conforming basis functions and results in matrix equation

$$P_M e_s + Q_M j = y_M \qquad (1.6)$$

Consequently, the CFIE with a parameter α can be determined by

$$P_\alpha e_s + Q_\alpha j_s = y_\alpha \qquad (1.7)$$

with

$$\begin{aligned} &P_\alpha = \alpha P_M + (1-\alpha) jkP_E;\ \mathrm{Q}_\alpha = \alpha Q_M + (1-\alpha) jkQ_E \\ &y_\alpha = \alpha y_M + (1-\alpha) jky_E \end{aligned} \qquad (1.8)$$

D. Final Matrix Equation

The final matrix equation for the FEM/IE formulation can be stated as

$$\begin{bmatrix} A & C \\ C^t & B - DQ^{-1}P \end{bmatrix}\begin{bmatrix} e_i \\ e_s \end{bmatrix} = \begin{bmatrix} 0 \\ DQ^{-1}y \end{bmatrix} (1.9)$$

and the matrix D is defined by

$$D_{ij} = \int_S \vec{w}_i \bullet \left(\hat{n}\times\vec{w}_j\right) dS \qquad (1.10)$$

where $\vec{w}_i$ are the curl-conforming vector basis functions. Moreover, the submatrices Q and P are obtained through EFIE, or MFIE, or CFIE.

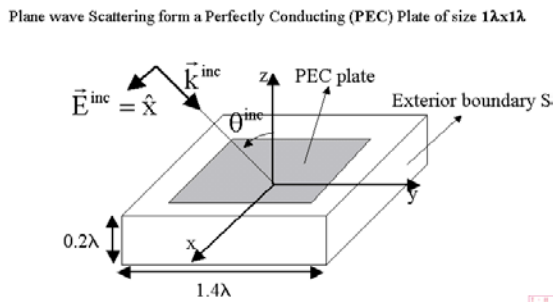
III. Preconditioner using p-type multiplicative Schwarz method

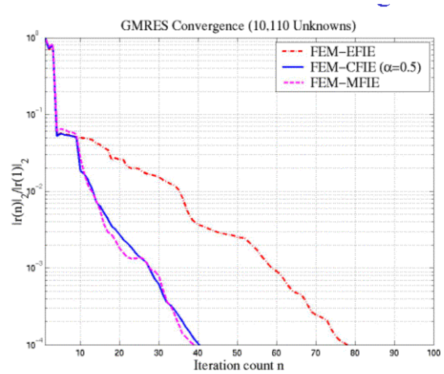
The preconditioner that is employed in this work is based on a multiplicative p-type Schwarz method. See reference [8].

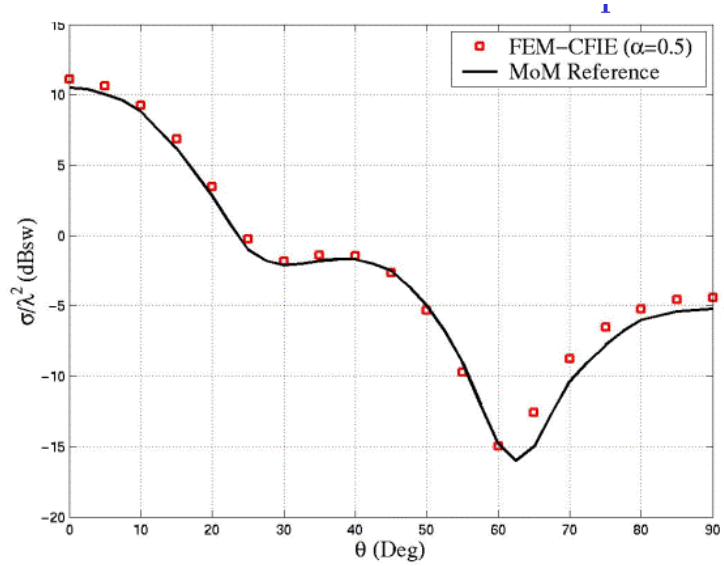
IV. Speed up by using a single-level low-rank approximation

The low-rank approximation algorithm is to extract the most dominant receivers and transmitters from both the receiver group and the transmitter group, respectively. The algorithm begins with the arbitrary selection of a receiver from the receiver group. With respect to this receiver, the most dominant transmitter is chosen from the transmitter group. This transmitter will then in turn determine the most dominant receiver, which is not the duplication of the chosen one. With these two chosen receivers, the second most dominant transmitter is selected. This process continues until no significant information is contributed by the remaining receivers or transmitters.

V. NUMERICAL RESULTS







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